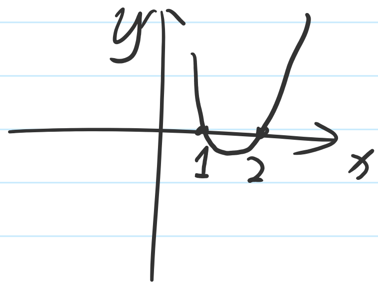
Chapter 5

Thursday, March 28, 2019 11:18 AM

Complex number.

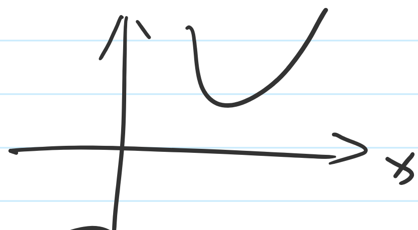
$(x-1)(x-2)=0$.

$x_1 = 1$, $x_2 = 2$.



$\boxed{x^2+x+1=0}$

$\Delta = b^2-4ac = 1^2-4\times1\times1<0$

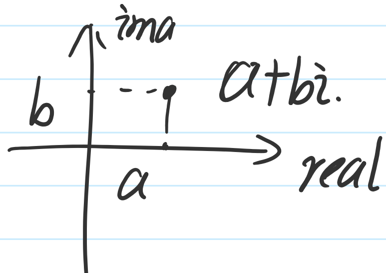


$x_1 = 1\square$ $x_2 = 1\square$.

$\sqrt{-1} = $ " i "

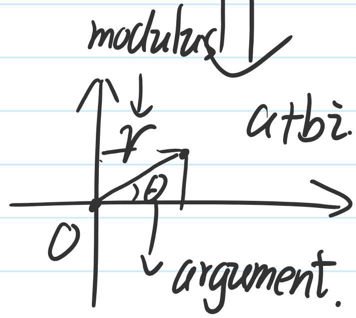
Def 1: $a+bi$

(I) $\underset{\text{real}}{a} + \underset{\text{imaginary}}{b}i$.



(II)

$r(\cos\theta + i\sin\theta)$



Polar form. $r\cdot(\cos\theta + i\sin\theta)$.

Euler formula

(III)

$\Downarrow$ Euler formula

$e^{i\theta} = \cos\theta + i\sin\theta$

(III)

$r \cdot e^{i\theta}$ Euler form $r \cdot e^{i\theta}$

P1.(b) $\dfrac{1-i}{3+2i} = \dfrac{(1-i)(3-2i)}{(3+2i)(3-2i)}$

$a+bi$

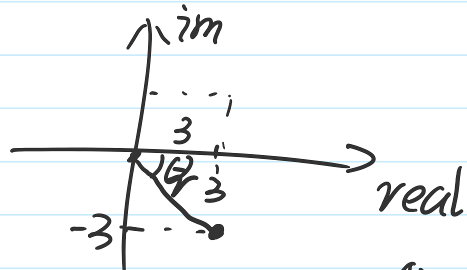
$a-bi$

$$= \frac{3-3i-2i+2\overset{-1}{(i)^2}}{9+\cancel{6i}-\cancel{6i}-\underbrace{(2i)^2}_{-4}}$$

$$= \frac{1-5i}{13} = \frac{1}{13} - \frac{5}{13}i$$

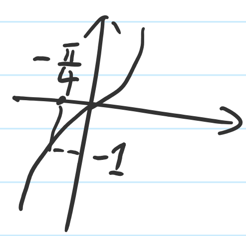
$a+bi$

P3:(a) $z_1 = 3-3i \longrightarrow$



r: $3\sqrt{2}$

θ: $\tan\theta = \dfrac{-3}{3} = -1 \Longrightarrow \theta = -\dfrac{\pi}{4}$

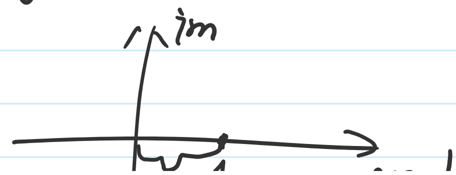


$z_1 = 3\sqrt{2}\left(\cos(-\frac{\pi}{4}) + i\sin(-\frac{\pi}{4})\right)$ Polar form

$\Downarrow$

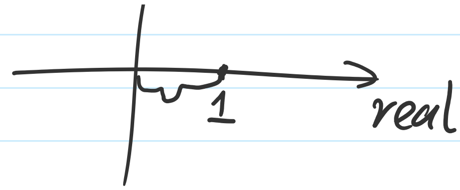
$z_1 = 3\sqrt{2}\, e^{i(-\frac{\pi}{4})}$ Euler form.

(j) $z_{10} = 1 - e^{i\frac{\pi}{4}}$



(j) $z_{10} = 1 - e^{\,\cdot}$

$= 1 \cdot e^{i0} - e^{i\frac{\pi}{4}}$



$r = 1, \quad \theta = 0$

$= \boxed{e^{i\left(\frac{0+\frac{\pi}{4}}{2}\right)}}\left(e^{-i\frac{\pi}{8}} - e^{i\frac{\pi}{8}}\right)$

$\Rightarrow \cos(-\frac{\pi}{8}) + i\sin(-\frac{\pi}{8})$

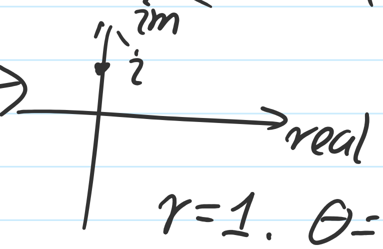
$\Rightarrow = \cancel{\cos(\frac{\pi}{8})} - i\sin(\frac{\pi}{8})$

$\Rightarrow \cancel{\cos(\frac{\pi}{8})} + i\sin(\frac{\pi}{8})$

$= e^{i\frac{\pi}{8}}\left(-2i\sin(\frac{\pi}{8})\right)$

$= -2\sin(\frac{\pi}{8})\, i\, e^{i\frac{\pi}{8}}$

$= -2\sin(\frac{\pi}{8}) \cdot 1 \cdot e^{i\frac{\pi}{2}} e^{i\frac{\pi}{8}}$



$r = 1. \quad \theta = \frac{\pi}{2}$

$= \underbrace{-2\sin(\frac{\pi}{8})}_{r}\, \underbrace{e^{i\frac{5\pi}{8}}}_{\theta = \frac{5\pi}{8}}$ Euler form

$\Longleftrightarrow -2\sin(\frac{\pi}{8})\left(\cos\frac{5\pi}{8} + i\sin\frac{5\pi}{8}\right)$. Polar form

P4. (d) $1 + \cos\theta - i\sin\theta$.

$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$ (A)

$\cancel{\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}}$ (B)

$= 1 + \frac{e^{i\theta} + e^{-i\theta}}{2} - \cancel{i}\,\frac{e^{i\theta} - e^{-i\theta}}{2\cancel{i}}$

$e^{i\theta} = \cos\theta + i\sin\theta$ ①

$= 1 + e^{-i\theta}$

$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta)$

$= \cos\theta - i\sin\theta$ ②

$= e^{i0} + e^{-i\theta}$

$= \boxed{e^{i\frac{0-\theta}{2}}}\left(e^{i\frac{\theta}{2}} + e^{i(-\frac{\theta}{2})}\right)$

$\frac{① + ②}{2} = \cos\theta \Rightarrow (A)$

$$= e^{i\frac{0-\theta}{2}}\left(e^{i\frac{\theta}{2}} + e^{i(-\frac{\theta}{2})}\right)$$

$$\frac{①+②}{2} = \cos\theta \Rightarrow (A)$$

$$= e^{-i\frac{\theta}{2}} \cdot 2\cos\frac{\theta}{2}$$

$$\frac{①-②}{2i} = \sin\theta \Rightarrow (B).$$

$$= 2\cos\frac{\theta}{2} \cdot e^{-i\frac{\theta}{2}} \longrightarrow \text{Euler form.}$$

$$\Longleftrightarrow 2\cos\frac{\theta}{2}\left(\cos\left(-\frac{\theta}{2}\right) + i\sin\left(-\frac{\theta}{2}\right)\right) \longrightarrow \text{Polar form.}$$

P5: (c) $z_3 = \left(\frac{(1-i)(\sqrt{3}+i)}{2i}\right)^{\boxed{12}}$

$$= \left(\frac{\sqrt{2}\left(\cos(-\frac{\pi}{4}) + i\sin(-\frac{\pi}{4})\right)\left(2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right)}{2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)}\right)$$

$$= \left(\frac{2\sqrt{2}\left\{\cos\left(-\frac{\pi}{4}+\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{4}+\frac{\pi}{6}\right)\right\}}{2\left(\cos\left(\frac{\pi}{2}\right) + i\sin\frac{\pi}{2}\right)}\right)$$

$$= \sqrt{2}\left(\cos\left(-\frac{7\pi}{12}\right) + i\sin\left(-\frac{7\pi}{12}\right)\right)^{\boxed{12}}$$

$$= (\sqrt{2})^{12}\left(\underbrace{\cos(-7\pi)}_{(-1)} + i\underbrace{\sin(-7\pi)}_{0}\right)$$

$$= 64 \cdot (-1)$$

$$= -64.$$

$z_1 = a_1 + b_1 i$

$z_2 = a_2 + b_2 i.$

$z_1 + z_2$, $z_1 - z_2$

$z_1 \cdot z_2$, $\frac{z_1}{z_2}$

$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$\checkmark\ z_1 \cdot z_2 = r_1 r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$

$\checkmark\ \frac{z_1}{z_2} = \left(\frac{r_1}{r_2}\right)(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)).$

$\checkmark\ z = r(\cos\theta + i\sin\theta)$

$z^n = r^n(\cos n\theta + i\sin n\theta)$

(h) $z_8 = \sqrt[4]{1+e^{i\frac{\pi}{4}}}$ $\quad n=4 \quad 1+e^{i\frac{\pi}{4}}$

(h) $z_8 = \sqrt[4]{1+e^{i\frac{\pi}{4}}}$ $n=4$ $1+e^{i\frac{\pi}{4}}$

$= e^{i0} + e^{i\frac{\pi}{4}}$

$= e^{i\left(\frac{0+\frac{\pi}{4}}{2}\right)}\left(e^{-i\frac{\pi}{8}} + e^{i\frac{\pi}{8}}\right)$

$= e^{i\frac{\pi}{8}} \cdot \left(2\cos\frac{\pi}{8}\right)$

$= 2\cos\frac{\pi}{8}\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)$

$= \left(2\cos\frac{\pi}{8}\left(\cos\frac{\pi}{8}+i\sin\frac{\pi}{8}\right)\right)^{\frac{1}{4}}$

$= \left(2\cos\frac{\pi}{8}\right)^{\frac{1}{4}}\left(\cos\frac{\frac{\pi}{8}+2k\pi}{4} + i\sin\frac{\frac{\pi}{8}+2k\pi}{4}\right)$

$\boxed{(k=0,1,2,3)}$ #

$z^{\frac{1}{n}} = r^{\frac{1}{n}}\left(\cos\frac{\theta+2k\pi}{n} + i\sin\frac{\theta+2k\pi}{n}\right)$

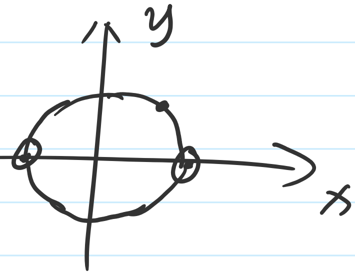
$\boxed{k=0, \cdots n-1}$ "n"

n-th root. "n" solutions.

P7. z $|z|=1$, $z \neq \pm 1$.

(a). $z_0 = \frac{1+z}{1-z}$. "purely imaginary"

$a + \boxed{bi}$, $a = 0$



$|z| = 1$. $\boxed{z} = r(\cos\theta + i\sin\theta)$, $r = 1$

$= \boxed{\cos\theta + i\sin\theta}$, $\theta \neq 0, \pi$.

$\boxed{z_0} = \frac{(1+\cos\theta)+i\sin\theta}{(1-\cos\theta)-i\sin\theta}$

$= \frac{\{[(1+\cos\theta)+i\sin\theta][(1-\cos\theta)+i\sin\theta]\}}{\{[(1-\cos\theta)-i\sin\theta][(1-\cos\theta)+i\sin\theta]\}}$

$\frac{a+bi}{c+di}$ rationalization.

$= (1-\cos^2\theta) + i^2\sin^2\theta + \ldots$

$$= \frac{(1-\cos^2\theta) + i^2\sin^2\theta + [\sin\theta(1-\cos\theta)+\sin\theta(1+\cos\theta)]i}{(1-\cos\theta)^2+\sin^2\theta}$$

$$= \frac{\cancel{\sin^2\theta} - \cancel{\sin^2\theta} + 2\sin\theta i}{2-2\cos\theta}$$

$$= \boxed{\frac{\sin\theta}{1-\cos\theta} i}$$ purely imaginary

P11: (c). $\boxed{z^{10} - 5z^5 - 6 = 0}$ $z^5 = y$

$y = z^5$

$\Rightarrow y^2 - 5y - 6 = 0$

$\Rightarrow (y-6)(y+1) = 0$

$\Rightarrow y_1 = 6, \quad y_2 = -1.$

$\Rightarrow z^5 = 6, \quad z^5 = -1.$

$\Rightarrow z = 6^{\frac{1}{5}}, \quad z = (-1)^{\frac{1}{5}}$ n-th roots

$= (6(\cos 0 + i\sin 0))^{\frac{1}{5}}$ n=5

$= 6^{\frac{1}{5}}(\cos\frac{0+2k\pi}{5} + i\sin\frac{0+2k\pi}{5})$

$\boxed{k = 0, 1, 2, 3, 4}$

$= (\cos\pi + i\sin\pi)^{\frac{1}{5}}$ n=5

$= 1^{\frac{1}{5}}(\cos\frac{\pi+2k\pi}{5} + i\sin\frac{\pi+2k\pi}{5})$

$\boxed{k = 0, 1, 2, 3, 4}$

(e) $\frac{z^5}{1+z^5} = \sqrt{3}i.$

(e) $\frac{z^5}{1+z^5} = \sqrt{3}i.$

$\Rightarrow z^5 = \sqrt{3}i(1+z^5)$

$= \sqrt{3}i + \underline{z^5\sqrt{3}i}$

$\Rightarrow (1-\sqrt{3}i)z^5 = \sqrt{3}i$

$\Rightarrow \underline{z^5 = \frac{\sqrt{3}i}{1-\sqrt{3}i}} = \frac{\sqrt{3}(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})}{2(\cos(-\frac{\pi}{3}) + i\sin(-\frac{\pi}{3}))}$

$= (\frac{\sqrt{3}}{2}(\cos(\frac{\pi}{2}+\frac{\pi}{3}) + i\sin(\frac{\pi}{2}+\frac{\pi}{3}))$

$z \quad = \left(\frac{\sqrt{3}}{2}(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6})\right)^{\frac{1}{5}}$

$\Rightarrow z = (\frac{\sqrt{3}}{2})^{\frac{1}{5}}\left(\cos(\frac{\frac{5\pi}{6}+2k\pi}{5}) + i\sin(\frac{\frac{5\pi}{6}+2k\pi}{5})\right)$

$n=5$ $\boxed{k=0,\dots,4}$

PD: $\boxed{z^4 - 8z^3 + 27z^2 - 50z + 50 = 0}.$

$\underbrace{3+i}_{a_1}$ solution $\Leftrightarrow$ $\underbrace{(3-i)}_{a_2}$ also solution

$(z-(3+i))(z-(3-i))(z-a_3)(z-a_4) = 0$

$(z^2-6z+10)(z^2-2z+5) = 0$

$\cancel{(z^2-6z+10)}$ $\quad z^2-2z+5 = 0$

$\cancel{(z^2-6z+10)}$ $\quad z^2-2z+5=0$

$$z_{1,2}=\frac{2\pm\sqrt{(-2)^2-4(1)(5)}}{2}$$

$$=1\pm 2i. \quad a_3, a_4.$$

$e^{i\theta}$ $\quad a + bi$

P3. (a) $\boxed{(\cos\theta+i\sin\theta)^5} = \boxed{\cos 5\theta + i\sin 5\theta}$

$\Rightarrow$ show $\boxed{\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta.}$ $\quad \sin 5\theta = ?$

binomial formula $\quad (a+b)^n=\sum_{k=0}^{n} a^k \cdot b^{n-k}$

$$=\boxed{\sum_{k=0}^{5}\binom{n}{k}\cos^k\theta\,(i\sin\theta)^{5-k}}$$

$k=0,2,4 \quad im$

$k=1 \quad k=3 \quad k=5$

$$\cdots\binom{5}{1}\cos\theta\cdot(i\sin\theta)^4+\binom{5}{3}\cos^3\theta\,(i\sin\theta)^2+\binom{5}{5}\cos^5\theta\,(i\sin\theta)^0$$

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta \quad \#.$$

Chapter 6

Thursday, April 11, 2019 10:58 AM

Matrix and Determinant

"+" "−" $A+B=\begin{pmatrix} a_{11}\pm b_{11} & & \\ & \ddots & \\ & & a_{nn}\pm b_{nn} \end{pmatrix}$ ✓

"×" $A_{a\times n}B_{n\times b}=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}_{2\times2}\cdot\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{pmatrix}_{2\times2}$

$=C_{a\times b}$

$=C=\begin{pmatrix} \boxed{c_{11}} & \square \\ \square & \square \end{pmatrix}_{2\times2}$

$c_{11}=a_{11}\cdot b_{11}+a_{12}\cdot b_{21}$

P1. (b). $A=\begin{pmatrix} 2 & -2 & 1 \\ 0 & 1 & 2 \\ 1 & -1 & 3\end{pmatrix}$ $B=\begin{pmatrix} 4 & -1 & 2 \\ 2 & -1 & 1 \\ 0 & 1 & 3\end{pmatrix}$.

$A(\underbrace{B+I_3}_{=})=A\cdot\begin{pmatrix} 4+1 & -1 & 2 \\ 2 & -1+1 & 1 \\ 0 & 1 & 3+1\end{pmatrix}$

$\begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1\end{pmatrix}_{3\times3}$ $=\begin{pmatrix} 2 & -2 & 1 \\ 0 & 1 & 2 \\ 1 & -1 & 3\end{pmatrix}_{3\times3}\begin{pmatrix} 5 & -1 & 2 \\ 2 & 0 & 1 \\ 0 & 1 & 4\end{pmatrix}_{3\times3}$

"1" number

"I_3" $A_{3\times3}=A_{3\times3}$. $=\begin{pmatrix} \square & \square & \square \\ \checkmark & \checkmark & \checkmark \\ \checkmark & \checkmark & \checkmark\end{pmatrix}=C$

$$C_{11} = 2\times5 + (-2)\times2 + 1\times0 = 6.$$

$$B = \begin{pmatrix} 2 & & 0 \\ & -3 & \\ 0 & & 1 \end{pmatrix}_{n\times n}$$ diagonal matrix.

"square matrix"

$$\boxed{B^n} = \begin{pmatrix} 2^n & & 0 \\ & (-3)^n & \\ 0 & & 1^n \end{pmatrix}$$

P4: A $\quad$ $\boxed{A^T}$ $\quad$ $-A^T$

$$\begin{pmatrix} a_{11} & \boxed{a_{12}} \\ \boxed{a_{21}} & a_{22} \end{pmatrix}_{2\times2}. \quad \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}_{2\times2} \quad \begin{pmatrix} -a_{11} & -a_{21} \\ -a_{12} & -a_{22} \end{pmatrix}.$$

$A = A^T$: symmetric $\Rightarrow \begin{cases} a_{11} = a_{11} \\ a_{22} = a_{22} \\ \boxed{a_{12} = a_{21}} \end{cases}$

$A = -A^T$: skew-symmetric $\Rightarrow \begin{cases} a_{11} = -a_{11} \\ a_{22} = -a_{22} \\ a_{12} = -a_{21} \end{cases} \Rightarrow \begin{cases} a_{11} = 0 \\ a_{22} = 0 \end{cases}$ ✓

"Determinant" number

$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}_{2\times2}$ $\quad |A| = \det A$ $\quad a_{11}\cdot a_{22} - a_{21}\cdot a_{12}$

$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}_{2\times 2}$. $|A| \triangleq \det A$ $\quad a_{11}\cdot a_{22} - a_{21}\cdot a_{12}$.

Expand $\det A$ along 1^{st} row:

$\det A = a_{11} A_{11} + a_{12} A_{12}$.

A_{ij} cofactor of a_{11}

$A_{ij} = (-1)^{i+j} \underbrace{M_{ij}}_{\text{Minor of } a_{ij}}$ $\Longrightarrow$

a_{ij} $\quad (m-1)\times(n-1)$.

$A_{m\times n}$

Minor of a_{ij}.

Pt. $A = \begin{pmatrix} 2 & 0 & -3 \\ 1 & 5 & 1 \\ 0 & 0 & 3 \end{pmatrix}_{3\times 3}$

(1) along 1^{st} row.

$30 \quad 0 \quad (-3)\cdot 0$

$\det A = a_{11}\cdot A_{11} + a_{12} A_{12} + a_{13} A_{13}$ $= \boxed{30}$.

$= 2\cdot(-1)^{1+1} M_{11} \quad 0\cdot \cancel{A_{12}} \quad (-3)\cdot(-1)^{1+3}\cdot M_{13}$

$= 2\cdot(-1)^2\cdot 15$

$\begin{vmatrix} 2 & 0 & -3 \\ 1 & 5 & 1 \\ 0 & 0 & 3 \end{vmatrix}$ $\qquad$ $\begin{vmatrix} 2 & 0 & -3 \\ 1 & 5 & 1 \\ 0 & 0 & 3 \end{vmatrix}$

$3\times 5 - 0\times 1$ $\qquad$ $= 0$

$= 15$.

$$3\times 5-0\times 1 = 15.$$

$\dot{0}$ 1

(2) along 2^{nd} column.

$$\det A = \begin{pmatrix} 2 & 0 & -3 \\ 1 & 5 & 1 \\ 0 & 0 & 3 \end{pmatrix}_{3\times 3} = \cancel{\underset{\overset{\parallel}{0}}{a_{12}}A_{12}} + \underset{\overset{\parallel}{5\cdot(-1)^{2+2}\cdot 6}}{a_{22}}\,A_{22} + \cancel{\underset{\overset{\parallel}{0}}{a_{32}}A_{32}}$$

$$= \boxed{30}$$

$$A_{22} = (-1)^{2+2} M_{22}$$

$$M_{22}: \begin{vmatrix} 2 & 0 & -3 \\ 1 & 5 & 1 \\ 0 & 0 & 3 \end{vmatrix} \Rightarrow \begin{vmatrix} 2 & -3 \\ 0 & 3 \end{vmatrix} = 6$$

P6. (d)

$$\det\begin{pmatrix} 1 & 0 & -4 \\ 2 & 1 & 1 \\ 1 & 0 & 5 \end{pmatrix}^{5}$$

① ②

$$\det(AB) = \det A \cdot \det B \qquad (B = A)$$

$$\boxed{\det(A^n) = (\det A)^n.}$$

$$= \Big(\underbrace{\det\begin{pmatrix} 1 & 0 & -4 \\ 2 & 1 & 1 \\ 1 & 0 & 5 \end{pmatrix}}_{\text{real number}}\Big)^5$$

$$= \Big(\cancel{\underset{\overset{\parallel}{0}}{a_{12}}A_{11}} + \underbrace{a_{22}}_{1}\,\underbrace{A_{22}}_{(-1)^{2+2}M_{22}} + \cancel{\underset{\overset{\parallel}{0}}{a_{32}}A_{32}}\Big)^5$$

$1 \quad (-1)^{2+2}|V|_{22}$

$$\begin{vmatrix} 1 & -4 \\ 1 & 5 \end{vmatrix} = 9$$

$= 9^5$

$= 59049.$

P7. $D = \begin{pmatrix} a & & 0 \\ & b & \\ 0 & & c \end{pmatrix}$ $\quad \det D = abc.$

$D_1 = \begin{pmatrix} a & d_1 & d_2 \\ & b & d_3 \\ 0 & & c \end{pmatrix}$ $\quad \det D_1 = abc$ ✓

P8. $\underset{4\times 4}{} \begin{matrix} A \\ B \end{matrix}$ $\quad \det A = 3, \quad \det B = 1.$

(a) $\det(A^3) = (\det A)^3 = 3^3 = 27$

(b) $\det(A^{-1}) = \dfrac{1}{\det A} = \dfrac{1}{3}$

$1 = \det I_n = \det(AA^{-1}) = \det A \cdot \det(A^{-1})$

$\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$

$$\boxed{\det(A^{-1}) = \frac{1}{\det A}}$$

$(\quad 1)$ $\boxed{\det(A_{\ }) = \det A}$

(c) $\det A^{-1}B = \det(A^{-1}) \cdot \det B = \frac{1}{3} \cdot 1 = \frac{1}{3}.$

(d) $\det B^T A = \det B^T \cdot \det A = 1 \cdot 3 = 3.$

$\boxed{\det B = \det B^T}$

(e) $\boxed{\det(\boxed{2} A_{n \times n})}$

$\boxed{k} A = \begin{pmatrix} ka_{11} & ka_{12} & \\ ka_{21} & \ddots & \\ & & ka_{mn} \end{pmatrix}$

$= 2^{n} \det A$

$\det A$

A: square matrix

$\det(kA) = k^{n} \det A$

$= 2^4 \cdot 3$

$= 48.$

$\begin{vmatrix} k a_{11} & ka_{12} \\ k a_{21} & ka_{22} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} \\ ka_{21} & ka_{22} \end{vmatrix}$

$= k^2 \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

(Inverse of matrix)

$ax = b$

$x = \frac{b}{a} = b \cdot a^{-1}$

$A^{-1}Ax = A^{-1}B \Rightarrow \boxed{x = A^{-1} \cdot B} \neq B \cdot A^{-1}$

$AB \neq BA$

P9 (a) $A = \begin{pmatrix} 2 & 0 & -1 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$

Step (i)

$\det A = \det \begin{pmatrix} 2 & 0 & -1 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix} \overset{2C_3 + C_1}{=} \det \begin{pmatrix} 0 & 0 & -1 \\ 3 & 2 & 1 \\ 6 & 1 & 3 \end{pmatrix}$

"Make hole"

॥

"0"

$= a_{13} A_{13}$

$= (-1) \cdot (-1)^{1+3} \cdot \begin{vmatrix} 3 & 2 \\ \cdot & \cdot \end{vmatrix}$

$$\parallel$$

"0"

$$= (-1)\cdot(-1)^{1+3}\cdot\begin{vmatrix}3 & 2\\ 6 & 1\end{vmatrix}$$
$$= (-1)\cdot(-9)$$
$$= 9 \neq 0$$

Step (ii)

$$A^{-1} = \frac{1}{\det A}\begin{pmatrix}A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{pmatrix}^T = \frac{1}{9}\begin{pmatrix}5 & -3 & 1\\ -1 & 6 & -2\\ 2 & -3 & 4\end{pmatrix}^T = \begin{pmatrix}\frac{5}{9} & -\frac{1}{9} & \frac{2}{9}\\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3}\\ \frac{1}{9} & -\frac{2}{9} & \frac{4}{9}\end{pmatrix}$$

Ex: $A_{11} = (-1)^{1+1}\begin{vmatrix}2 & 1\\ 1 & 3\end{vmatrix} = 5 \quad \ldots$

$A_{12} = \ldots$

$A_{13} = \ldots$

(System of linear Equations)

P14 (g)
$$\begin{cases} x+2y+3z+4w=-2\\ 2x+4y+5z+9w=1\\ -3x-6y \quad\quad +w=4\end{cases} \Rightarrow \begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 4 & 5 & 9\\ -3 & -6 & 0 & 1\end{pmatrix}_{3\times4}\begin{pmatrix}x\\ y\\ z\\ w\end{pmatrix}_{4\times1} = \begin{pmatrix}-2\\ 1\\ 4\end{pmatrix}$$

$$\xrightarrow[\text{Augmented matrix}]{\text{Gaussian Elimination}} \left(\begin{array}{cccc|c}1 & 2 & 3 & 4 & -2\\ \boxed{2} & 4 & 5 & 9 & 1\\ \boxed{-3} & -6 & 0 & 1 & 4\end{array}\right) \xrightarrow[R_3+3R_1]{R_2-2R_1} \left(\begin{array}{cccc|c}1 & 2 & 3 & 4 & -2\\ 0 & 0 & -1 & 1 & 5\\ 0 & 0 & 9 & 13 & -1\end{array}\right)$$

$$\Big\Downarrow R_3+9R_2$$

$$\begin{cases} x+2y+3z+4w=-2\\ \quad\quad\quad -z+w=5\\ \quad\quad\quad\quad 22w=44.\end{cases} \Leftarrow \left(\begin{array}{cccc|c}1 & 2 & 3 & 4 & -2\\ 0 & 0 & -1 & 1 & 5\\ 0 & 0 & 0 & 22 & 44\end{array}\right)$$

$\Rightarrow w=2 \quad z=-3 \quad x+2y=4 \quad \ldots$

$-2w \quad +1.$

$\Rightarrow w=2, \; z=-3, \quad x+2y=-1.$

$\Updownarrow$ (free variable)

$y=t \Rightarrow x=-1-2t.$

TLQs

vector form $\Rightarrow$ $X = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -3 \\ 2 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

P18 Consider the system $\begin{cases} x-2y+z=1 \\ x-y+2z=2 \\ \quad\;\; y+c^2z=c. \end{cases}$

$\xrightarrow[\text{Matrix}]{\text{Augmented}} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 1 & -1 & 2 & 2 \\ 0 & 1 & c^2 & c \end{array}\right) \overset{R_2-R_1}{\Longrightarrow} \left(\begin{array}{ccc|c} -1 & -2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & c^2 & c \end{array}\right)$

$\Downarrow R_3-R_2$

$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & c^2-1 & c-1 \end{array}\right)$

(a) $c^2-1 \neq 0 \Rightarrow c \neq \pm 1.$

Unique solution.

(b) $\begin{cases} c^2-1=0 \\ c-1=0 \end{cases} \Rightarrow c=1.$

Infinitely many solutions.

(c) $\begin{cases} c^2-1=0 \\ c-1 \neq 0 \end{cases} \Rightarrow c=-1.$

NO solution.

NO solution.

Chapter 1

Monday, February 18, 2019 12:39 PM

$$F'(x) = f(x) \Longleftrightarrow \boxed{\int f(x)\,dx = F(x) + C}$$

intrand primitive function

P1:

(a) $\int \cos(3x+1)\,dx$

$= \frac{1}{3}\int \cos(3x+1)\,d(3x+1)$

$\overset{y=3x+1}{=} \frac{1}{3}\int \cos y\,dy$

$= \frac{1}{3}\sin x + C$

$$\boxed{\int \cos x\,dx = \sin x}$$

substitution

$$\times\ \boxed{\int f(ax+b)\,dx = \frac{1}{a}F(ax+b)}$$

(d) $\int \frac{1}{1+16x^2}\,dx$

$= \int \frac{1}{1+(4x)^2}\,dx$

$= \frac{1}{4}\int \frac{1}{1+(4x)^2}\,d(4x)$

$\overset{y=4x}{=} \frac{1}{4}\int \frac{1}{1+y^2}\,dy$

$= \frac{1}{4}\tan^{-1}(y) + C$

$= \frac{1}{4}\tan^{-1}(4x) + C$

$$\boxed{\int \frac{1}{1+x^2}\,dx = \tan^{-1}(x) + C.}$$

brief table

$\int \frac{1}{x^2}\,dx = \int x^{-2}\,dx = -1\cdot x^{-1}$

$\int \frac{1}{x}\,dx = \ln|x| + C$

$$\boxed{\int x^a\,dx = \frac{1}{a+1}x^{a+1} + C.}$$

P2: (a) $\int \frac{x^2-x+1}{x^2}\,dx = \int\left(1-\frac{1}{x}+\frac{1}{x^2}\right)dx = x - \ln|x| + (-1)x^{-1}$

P2 (a) $\int \frac{x^2 - x + 1}{x^2} dx = \int (1 - \frac{1}{x} + \frac{1}{x^2}) dx = x - \ln|x| + (-1)x^{-1} + C$

$= x - \ln|x| - \frac{1}{x} + C$

(b) $\int \frac{2x^2}{x^2+1} dx = \int \frac{2(x^2+1)-2}{x^2+1} dx = \int (2 - \frac{2}{x^2+1}) dx = 2x - 2\int \frac{1}{x^2+1} dx$

$= 2x - 2\tan^{-1}(x) + C.$

P3 (a). $\int_1^2 \frac{x-1}{3x^2} dx$

$= \int_1^2 (\frac{x}{3x^2} - \frac{1}{3x^2}) dx = \int_1^2 (\frac{1}{3x} - \frac{1}{3x^2}) dx$

$= \frac{1}{3} \int_1^2 (\frac{1}{x} - \frac{1}{x^2}) dx.$

$= \frac{1}{3} \left(\ln|x| \Big|_1^2 - (-1) \cdot x^{-1} \Big|_1^2 \right)$

$= \frac{1}{3} (\ln 2 - ((-1) \cdot 2^{-1} - (-1) \cdot 1^{-1}))$

$= \frac{1}{3} \ln 2 - \frac{1}{6}.$

$\int_1^2 x^{-2} dx$

$= (-1) \cdot x^{-1} \Big|_1^2$

$((-1) \cdot x^{-1})'$

$= (-1) \cdot (-1) \cdot x^{-2}$

$= x^{-2}$

P2. (g) $\int \frac{3}{x^2 - 2x + 5} dx = \int \frac{3}{(x-1)^2 + 4} dx$

$\Longrightarrow \int \frac{1}{x^2+1} dx = \tan^{-1}(x) + C$

$= 3 \int \frac{\frac{1}{4}}{(\frac{x-1}{2})^2 + 1} dx$

$= \frac{3}{4} \int \frac{1}{(\frac{x-1}{2})^2 + 1} dx$

$= \frac{3}{4} \int 2 \frac{1}{(\frac{x-1}{2})^2 + 1} d(\frac{x-1}{2})$

$\overset{y = \frac{x-1}{2}}{=} \frac{3}{2} \int \frac{1}{y^2+1} dy$

$= \frac{3}{2} \tan^{-1}(y) + C = \frac{3}{2} \tan^{-1}(\frac{x-1}{2}) + C$

$$= \frac{3}{2}\tan^{-1}(y) + C = \frac{3}{2}\tan^{-1}\left(\frac{x-1}{2}\right) + C$$ #.

(i) $\int \frac{x+6}{(2x-1)^3}\,dx = \int \frac{\frac{1}{2}(2x-1)+\frac{13}{2}}{(2x-1)^3}\,dx$.

$$= \int \left(\frac{1}{2}\cdot\frac{1}{(2x-1)^2} + \frac{13}{2}\cdot\frac{1}{(2x-1)^3}\right)dx$$

$$= \frac{1}{2}\int \frac{1}{(2x-1)^2}\,dx + \frac{13}{2}\int \frac{1}{(2x-1)^3}\,dx.$$

$$= \frac{1}{2}\int \frac{1}{2}\frac{1}{(2x-1)^2}\,d(2x-1) + \frac{13}{2}\int \frac{1}{2}\frac{1}{(2x-1)^3}\,d(2x-1)$$

$\int \frac{1}{x^2}dx = \int x^{-2}dx$

$\int \frac{1}{x^3}dx = \int x^{-3}dx$

$y = 2x-1$

$$= \frac{1}{4}\int \frac{1}{y^2}\,dy + \frac{13}{4}\int \frac{1}{y^3}\,dy.$$

$$= \frac{1}{4}(-1)y^{-1} + \frac{13}{4}\cdot\left(-\frac{1}{2}\right)y^{-2} + C$$

$$= -\frac{1}{4y} - \frac{13}{8}\frac{1}{y^2} + C$$

$$= -\frac{1}{4(2x-1)} - \frac{13}{8}\frac{1}{(2x-1)^2} + C.$$ #.

P1 (d) $\int \sin 3x \sin 2x\,dx \overset{\text{product}\Rightarrow\text{sum}}{=\!=} \int -\frac{1}{2}\left[\cos(3x+2x) - \cos(3x-2x)\right]dx$

$$= -\frac{1}{10}\sin 5x + \frac{1}{2}\sin x + C$$

(f) $\int \frac{1}{(x-1)(2x-3)}\,dx.$

$$\frac{1}{(Ax+b)(Bx+C)} = \frac{C}{Ax+b} + \frac{C}{Bx+C}$$

$$= \int\left(\frac{-1}{x-1} + \frac{2}{2x-3}\right)dx \qquad \frac{A}{x-1} + \frac{B}{2x-3} = \frac{A(2x-3)+B(x-1)}{(x-1)(2x-3)} = 1$$

$$= -1\ln|x-1| + \ln|2x-3| + C$$

put $x = \frac{3}{2}$ $B = 2$

$= -1 \ln|x-1| + \ln|2x-3| + C$

$\int \frac{2}{2x-3} dx = \int 2 \cdot \frac{1}{2} \frac{1}{2x-3} d(2x-3)$ $\quad d(2x-3) = 2$

$= \int \frac{1}{2x-3} d(2x-3)$

$\overset{y=2x-3}{=} \ln|2x-3|$ #.

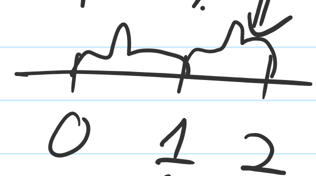
$(2x-3)$

put $x = \frac{3}{2}$, $B=2$

put $x=1$, $A=-1$.

P3: (g) $\int_0^2 e^{1+|x-1|} dx$.

$|x-1| = -(x-1)$ $\quad |x-1| = x-1$



$$|x-1| = \begin{cases} x-1, & 1 \le x \le 2 \\ -(x-1), & 0 \le x < 1. \end{cases}$$

$$\boxed{\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx}$$

$= \int_0^1 e^{1-(x-1)} dx + \int_1^2 e^{1+x-1} dx.$

$= \int_0^1 e^{2-x} dx + \int_1^2 e^x dx.$

$= \int_0^1 (-1) e^{2-x} d(2-x) + e^x \Big|_1^2.$ $\quad dx = -1\, d(2-x)$

$x=0$, $y=2-x=2$.

$x=1$, $y=2-1=1$.

$\overset{y=2-x}{=} \int_2^1 (-1) \cdot e^y dy + e^2 - e^1$

$= -1 e^y \Big|_2^1 + e^2 - e^1.$

$= -1 \cdot e^1 - (-1) \cdot e^2 + e^2 - e^1.$

$= 2e^2 - 2e^1$ #.

21/02/2019. Thursday 11:00 am – 12:00am

P4: $\frac{d}{dx}\int_3^x e^{2y^2+1}\,dy$

$\int e^{2y^2+1}\,dy = F(y)$

$f(x)$

$= \frac{d}{dx}\left(F(x) - F(3)\right)$ → number

$= \frac{dF(x)}{dx} - 0$

$= e^{2x^2+1} - 0 = e^{2x^2+1}$

P5 (a): $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

$F(a) - F(0)$

$-F(a-x)\Big|_0^a$

$F(a) - F(0)$

$\int f(x)\,dx = F(x)$

$\int f(ax+b)\,dx = \frac{1}{a}F(ax+b)$

Feedback of assignment 1

Monday, February 25, 2019 1:24 PM

P1:

1. (10 points) Find the point of intersection of the lines

$$l_1: \textcircled{1}\begin{cases} x = t \\ y = -t+2 \\ z = t+1 \end{cases} \quad \text{and} \quad l_2: \textcircled{2}\begin{cases} x = 2s+2 \\ y = s+3 \\ z = 5s+6 \end{cases},$$

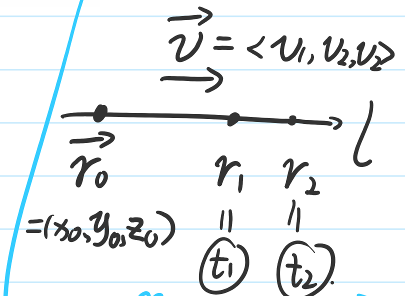
and then find the equation of the plane determined by these lines.

Pf: Recall that the vector equation for a line L passing through a point $P_0(x_0,y_0,z_0)$ parallel to a vector $\vec v = \langle v_1, v_2, v_3\rangle$ is

$$\vec r(t) = \vec r_0 + t\vec v, \qquad -\infty < t < \infty,$$

where $\vec r$ is the position vector of a point $P(x,y,z)$ on L and $\vec r_0$ is the position vector of $P_0(x_0,y_0,z_0)$. In component form, the vector equation is equivalent to three scalar equations:

$$\begin{cases} x = x_0 + tv_1 \\ y = y_0 + tv_2 \\ z = z_0 + tv_3 \end{cases}.$$

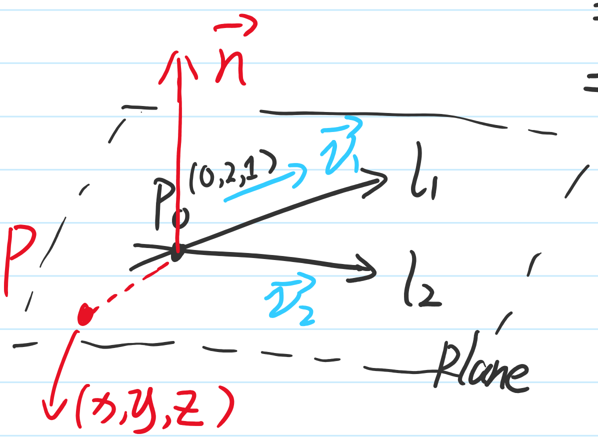


$$l_1 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \qquad (r_1,\ \vec v_1)$$

$$l_2 = \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} + s\begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} \qquad (r_2,\ \vec v_2)$$

Link ① and ② $\Rightarrow \begin{cases} t = 2s+2 \\ -t+2 = s+3 \\ t+1 = 5s+6 \end{cases} \Rightarrow \underbrace{t=0,\ s=-1}$ satisfy

Then intersected point $P_0 = (t, -t+2, t+1)|_{t=0}$
$= (2s+2, s+3, 5s+6)|_{s=-1}$
$= (0, 2, 1)$



$\vec v_1 = (1,-1,1) \quad \vec v_2 = (2,1,5)$

$$\vec n = \vec v_1 \times \vec v_2 = \begin{vmatrix} \vec i & \vec j & \vec k \\ 1 & -1 & 1 \\ 2 & 1 & 5 \end{vmatrix} = -6\vec i - 3\vec j + 3\vec k.$$

$(-6, -3, 3)$

$\vec{P_0P} = (x, y-2, z-1)$

$\underline{\vec n \perp \vec{P_0P}\ ?} \Rightarrow \vec n \cdot \vec{PP_0} = 0$

$\Rightarrow (-6)(x-0) + (-3)(y-2) + 3(z-1) = 0$

$\Rightarrow -6x - 3y + 3z = -3.$

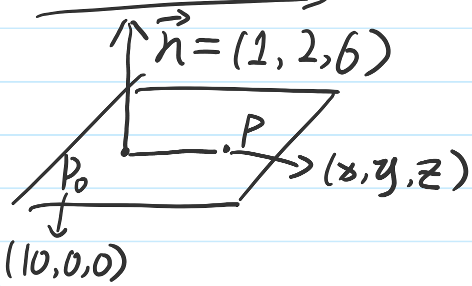
P2

2. (15 points) Find the distance from the line

$$\begin{cases} x = t+2 \\ y = t+1 \\ z = -\frac{1}{2}(t+1) \end{cases} \qquad \begin{pmatrix} 2 \\ 1 \\ -\frac{1}{2} \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ -\frac{1}{2} \end{pmatrix}$$

to the plane $x + 2y + 6z = 10$. (Hint: Explain first why the line is parallel with the plane (the normal vector of the plane is perpendicular to the direction vector of the line). Then pick an arbitrary point A on the line and an arbitrary point to the plane $x+2y+6z=10$. (Hint: Explain first why the line is parallel with the plane (the normal vector of the plane is perpendicular to the direction vector of the line). Then pick an arbitrary point A on the line and an arbitrary point B on the plane. Project the vector $\overrightarrow{AB}$ (orthogonally) onto the normal vector of the plane, and find the length of the projected vector.)

Pf: $1\cdot(x-10)+2\cdot y+6\cdot z=0$



(I) $\Rightarrow \vec{n}\cdot\overrightarrow{P_0P}=0 \Rightarrow \vec{n}\perp \overrightarrow{P_0P}$.

(II) Find three points on this plane

$A(10,0,0)$ $B(0,5,0)$ $C(0,0,\frac{10}{6})$

$\overrightarrow{AB}=(-10,5,0)$ $\overrightarrow{AC}=(10,0,\frac{10}{6})$

$\vec{n}=\overrightarrow{AB}\times\overrightarrow{AC}$.

$\Leftarrow$ How to find normal vector for a plane?

$\Rightarrow \vec{n}=(1,2,6)$ $\vec{v}=(1,1,-\frac{1}{2})$

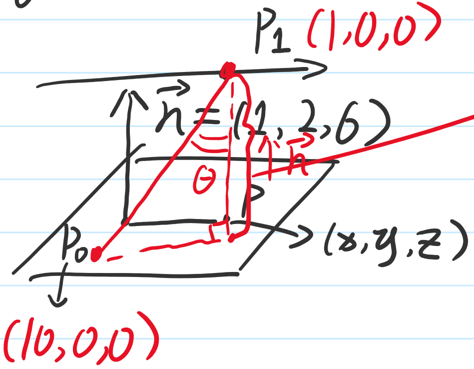
$\Rightarrow \vec{n}\cdot\vec{v}=1\times1+2\times1+6\times(-\frac{1}{2})=0 \Rightarrow \vec{n}\perp\vec{v}$.

Then select random point in the line and plane.

$P_1=(t+2,\ t+1,\ -\frac{1}{2}(t+1))\big|_{t=-1}=(1,0,0)$

$P_0=(10,0,0)$

$\overrightarrow{P_1P_0}=(9,0,0)$ $\vec{n}=(1,2,6)$



$d=\left|\mathrm{Proj}_{\vec{n}}\overrightarrow{P_1P_0}\right|$

$=\left|\overrightarrow{P_1P_0}\cdot\cos\theta\right|$

$=\left|\,|\overrightarrow{P_1P_0}|\cdot\dfrac{\overrightarrow{P_1P_0}\cdot\vec{n}}{|\overrightarrow{P_1P_0}|\cdot|\vec{n}|}\right|$

$$= \left| |P_1P_0| \cdot \frac{}{|P_1P_0| \cdot |\vec{n}|} \right|$$

$$= \frac{9}{\sqrt{41}}.$$

Chapter 2

Monday, February 25, 2019 1:52 PM

$$\begin{cases} \text{Substitution Method.} \\ \cdots \text{ by part.} \end{cases}$$

P1. (a). $\int \frac{e^{1+\frac{1}{x^2}}}{x^3} dx$

$y = 1 + \frac{1}{x^2}$

$\Downarrow$

$dy = (-2) x^{-3} dx$

$\Downarrow$

$dx = (-\frac{1}{2}) x^3 dy$

$$\int \frac{e^{y}}{\cancel{x^3}} \cdot (-\tfrac{1}{2}) \cancel{x^3} dy$$

$$= (-\tfrac{1}{2}) \int e^{y} dy$$

$$= (-\tfrac{1}{2}) e^{y} + C.$$

$$= (-\tfrac{1}{2}) e^{1+\frac{1}{x^2}} + C.$$

(e). $\int \sin 2x \sqrt{\cos x}\, dx$

$y = \cos x$

$\Downarrow$

$dy = -\sin x\, dx$

$\Downarrow$

$dx = -\frac{1}{\sin x} dy$

$$= \int \sin 2x \cdot \sqrt{y} \left(-\frac{1}{\sin x}\right) dy$$

$$= \int 2 \cancel{\sin x} \cos x \sqrt{y} \cdot \left(-\frac{1}{\cancel{\sin x}}\right) dy$$

$$= \int 2 \cdot y \cdot \sqrt{y}\, dy$$

$$= 2 \int y^{\frac{3}{2}} dy = -\frac{4}{5} (y)^{\frac{5}{2}} + C.$$

$$= -\frac{4}{5} (\cos x)^{\frac{5}{2}} + C.$$

look at (j)(k) (I) (II)

look at (j)(k)

(k) $\int \frac{4x+4-4}{3x^2+6x+19}dx = \int \frac{4x+4}{3x^2+6x+19}dx + \int \frac{-4}{3x^2+6x+19}dx$

(I) (II)

$\frac{1}{1+x^2}$

(I) $y = 3x^2+6x+19$

$\Downarrow$

$dy = (6x+6)dx$

$\Downarrow$

$dx = \frac{1}{6x+6}dy$

$\int \frac{4x+4}{y} \cdot \frac{1}{6x+6} dy = \frac{2}{3}\int \frac{1}{y}dy = \frac{2}{3}\ln|y| + C$

$= \frac{2}{3}\ln|3x^2+6x+19| + C.$

(II) $\int \frac{-4}{3x^2+6x+19}dx$

$\int \frac{1}{1+x^2}dx = \tan^{-1}(x)+C$

$= \int \frac{-4}{3(x+1)^2+16}dx = \int \frac{-\frac{1}{4}}{\frac{3}{16}(x+1)^2+1}dx$

$\Updownarrow$

$= (-\frac{1}{4})\int \frac{4}{\sqrt{3}} \frac{1}{(\frac{\sqrt{3}}{4}(x+1))^2+1} d(\frac{\sqrt{3}}{4}(x+1))$

$\frac{\sqrt{3}}{4}dx$

$= (-\frac{1}{\sqrt{3}}) \tan^{-1}(\frac{\sqrt{3}}{4}(x+1)) + C.$

Result = (I) + (II) $= \frac{2}{3}\ln|3x^2+6x+19| + (-\frac{1}{\sqrt{3}})\tan^{-1}(\frac{3}{4}(x+1)) + C.$

(l)(m)(n)

(l). $\int \frac{1}{x^2\sqrt{1-x^2}}dx$. $y = 1-x^2$ $\Downarrow$ $dy = -2x\,dx$ $\Downarrow$ $dx = -\frac{1}{2x}dy$

$\int \frac{1}{x^2\sqrt{y}} \cdot (-\frac{1}{2x})dy$

$= \cancel{\int \frac{1}{1-y} \cdot \frac{1}{\sqrt{y}} \cdot (-\frac{1}{2x})dy}$

$\sqrt{1-x^2} \overset{x=\sin\theta}{=\!=} \sqrt{1-\sin^2 x} = \cos x$

$\sqrt{1+cx^2} \overset{x=\frac{1}{\sqrt{c}}\tan\theta}{=\!=} \sqrt{1+c\ldots}$

$(\cos x)$ $\quad = \int \frac{1}{1-y}\sqrt{y}\,\left(-\frac{1}{2x}\right)dy.$

$$\sqrt{1+cx^2} \xlongequal{x=\frac{1}{\sqrt{c}}\tan\theta} \sqrt{1+c\cdot\frac{1}{c}\tan^2\theta} = \sqrt{\sec^2\theta} = \sec\theta$$

$$\Rightarrow x = \sin\theta \Rightarrow dx = \cos\theta\, d\theta \quad \Rightarrow \theta = \sin^{-1}(x)$$

$$\int \frac{1}{\sin^2\theta\cdot\cancel{\cos\theta}}\cdot\cancel{\cos\theta}\,d\theta = \int\frac{1}{\sin^2\theta}d\theta = -\cot\theta + C$$

$$= -\cot(\sin^{-1}(x)) + C$$

$$-\frac{\sqrt{1-x^2}}{x} + C \quad \#$$

(P). $\displaystyle\int \frac{1}{(x^2+6x+10)^{\frac{3}{2}}}dx.$

$$= \int\frac{1}{[(x+3)^2+1]^{\frac{3}{2}}}dx.$$

$x+3 = \tan\theta$ $\Downarrow$ $dx = \sec^2\theta\, d\theta$

$$\int\frac{1}{(1+\tan^2\theta)^{\frac{3}{2}}}dx$$

$$\int\frac{1}{\cancel{\sec^3\theta}}\,\cancel{\sec^2\theta}\,d\theta$$

$$1+\tan^2\theta = \frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} = \sec^2\theta.$$

$$(1+\tan^2\theta)^{\frac{3}{2}} = \sec^3\theta.$$

$$= \int\frac{1}{\sec\theta}d\theta$$

$$= \int\cos\theta\, d\theta \Rightarrow -\sin\theta + C.$$

(I). $\theta = \tan^{-1}(x+3)$

$\sin(\tan^{-1}(x+3)) + C.$ ✓

(II) $x+3$, $\sqrt{(x+3)^2+1}$, θ, 1

$$= -\frac{x+3}{\sqrt{(x+3)^2+1}} + C.$$

$\int v\,du = uv - \int u\,dv$

$\int u'v\,dx = uv - \int uv'\,dx$

P2. (Integration by part $\Leftarrow$) $\displaystyle\int (uv)'\,dx = \int u'v\,dx + \int uv'\,dx$

uv

(c) $\displaystyle\int x^2\sin x\,dx$ $\qquad$ (d) $\displaystyle\int x\sin^2 x\,dx.$

(c) $\int x^2 \sin x \, dx$

(I) $\int \sin x \, d\left(\frac{1}{3}x^3\right)$

uv

$\int x^2 \cdot \ln x \, dx$

$= \sin x \cdot \frac{1}{3}x^3 - \int \frac{1}{3}x^3 \, d\sin x$

$= \sin x \cdot \frac{1}{3}x^3 - \int \frac{1}{3}x^3 \cos x \, dx.$

(II) $\int x^2 \cdot d(-\cos x)$

$= x^2 \cdot (-\cos x) - \int -\cos x \, dx^2$

$= x^2 \cdot (-\cos x) + \int \cos x \cdot 2x \, dx.$ ①

$= x^2 \cdot (-\cos x) + 2\int x \cdot d\sin x$

$= x^2 \cdot (-\cos x) + 2x \cdot \sin x - \int \sin x \, dx$ ②

$= x^2 \cdot (-\cos x) + 2x \cdot \sin x + \cos x + C.$

(d) $\int x \sin^{\boxed{2}} x \, dx$

$(\_\_)' = \sin^2 x$

$= \int x \cdot \frac{1-\cos 2x}{2} dx$

$= \int \frac{1}{2}x - \frac{1}{2}x\cos 2x \, dx$

$= \int \frac{1}{2}x \, dx - \frac{1}{2}\int x \cdot \cos 2x \, dx$

$= \frac{1}{4}x^2 - \frac{1}{2} \cdot \frac{1}{2} \int x \, d\sin 2x.$

$= \frac{1}{4}x^2 - \frac{1}{2} \cdot \frac{1}{2}\left[x \cdot \sin 2x - \int \sin 2x \, dx\right]$

$= \frac{1}{4}x^2 - \frac{1}{4}x\sin 2x - \frac{1}{8}\cos 2x.$

#.

(g) $\int \csc^{\boxed{3}} x \, dx$

$= \int \csc^2 x \cdot \csc x \, dx$

$= \int \csc x \, d(-\cot x)$

$= \csc x \cdot (-\cot x) - \int -\cot x \, d(\csc x)$

$= \csc x \cdot (-\cot x) - \int \csc x \cdot \cot^2 x \, dx$

$= \boxed{\cdot} - \int \csc x \cdot (\csc^2 x - 1) \, dx$

(h) $\int \cos^3 x \, dx$

$\cos^2 x \cdot \cos x$

$\int \csc^2 x \, dx = -\cot x$

$(-\cot x)' = \csc^2 x$

$= -\csc x \cdot \cot x$

$\csc x = \frac{1}{\sin x}$

$$-\boxed{} - \int \csc x \cdot (\csc^2 x - 1)\,dx$$

$$= \boxed{} - \int \csc^3 x\,dx + \int \csc x\,dx$$

$$\int \csc^3 x\,dx = \boxed{} + \int \csc x\,dx \checkmark$$

$$\int \csc^3 x\,dx = \frac{1}{2}\left(-\csc x \cdot \cot x - \ln|\cot x + \csc x|\right) + C$$

P3 (a). $\int e^{2x} \sin(2e^x + 1)\,dx$

$$\boxed{y = 2e^x + 1} \Rightarrow dy = 2e^x dx \Rightarrow dx = \frac{1}{2e^x} dy$$

$$\int \cancel{e^{2x}} \sin y \cdot \frac{1}{2\cancel{e^x}} dy \qquad (e^x)^2$$

$$= \int e^x \frac{1}{2} \sin y\,dy$$

$$= \int \left(\frac{y-1}{2}\right) \frac{1}{2} \sin y\,dy$$

$$= \frac{1}{2}\int y \cdot \sin y\,dy - \frac{1}{2}\int \sin y\,dy$$

$$= \frac{1}{4}\int y\,d(-\cos y) + \frac{1}{4}\cos y$$

$$= -\frac{1}{4}\left(y \cdot \cos y - \int \cos y\,dy\right) + \frac{1}{4}\cos y$$

$$= -\frac{1}{4} y \cdot \cos y + \frac{1}{4}\sin y + \frac{1}{4}\cos y + C$$

$$= -\frac{1}{4}(2e^x+1)\cos(2e^x+1) + \frac{1}{4}\sin(2e^x+1) + \frac{1}{4}\cos(2e^x+1) + C$$

(C.) $\int_0^1 \ln(1+\sqrt[3]{x})\,dx$ when $x=0$, $y=1$

(C.) $\int_0^1 \ln(1+\sqrt[3]{x})\,dx$.

when $x=0$, $y=1$.
when $x=1$, $y=2$.

$y=1+x^{\frac{1}{3}}$ $\overset{\#}{=}$ $\int_1^2 \ln y \cdot 3x^{\frac{2}{3}}\,dy$

$dy=\frac{1}{3}x^{-\frac{2}{3}}dx$
$\Downarrow$
$dx=3x^{\frac{2}{3}}dy$ $\quad 3\int_1^2 \ln y\cdot(y-1)^2\,dy$

$(x^{\frac{1}{3}})^2=(y-1)^2$

$= \int_1^2 \ln y\, d(y-1)^3$

$= \ln y\cdot(y-1)^3\Big|_1^2 - \int_1^2 (y-1)^3\, d(\ln y)$

$= \ln 2 - 0 - \int_1^2 (y-1)^3\cdot\frac{1}{y}\,dy$.

$= \ln 2 - \int_1^2 \frac{y^3-3y^2+3y-1}{y}\,dy$.

$\ln y\Big|_1^2$

$= \ln 2 - \int_1^2 y^2\,dy + \int_1^2 3y\,dy - \int_1^2 3\,dy + \int_1^2 \frac{1}{y}\,dy$

$= -\frac{5}{6}$

II

$(\ )' = \ln x$

$\int \ln x\cdot x\,dx$

$= \int \ln x \cdot d(\frac{1}{2}x^2)$

$= \ln x\cdot(\frac{1}{2}x^2) - \int \frac{1}{2}x^2\, d\ln x$

$= \ln x\cdot(\frac{1}{2}x^2) - \int \frac{1}{2}x^{\cancel{2}}\cdot\frac{1}{\cancel{x}}\,dx$

$= \ln x\cdot(\frac{1}{2}x^2) - \frac{1}{2}\int x\,dx$

$= \ln x\cdot(\frac{1}{2}x^2) - \frac{1}{4}x^2$ #.

(i). $\int x^2\sqrt{4-x^2}\,dx$. $\quad y=4-x^2$ $\overset{\#}{=}$ $\int x^{2}\sqrt{y}\cdot\frac{-1}{2x}\,dy$

$x^2=4-y. \Rightarrow x=\sqrt{4-y}$

$dy=-2x\,dx$
$\Downarrow$
$dx=-\frac{1}{2x}dy$

$= \int x\sqrt{y}\,dy$

$= \int \sqrt{4-y}\cdot\sqrt{y}\,dy$

$\left(\begin{array}{l} \sqrt{c+cx^2} \quad x=\tan\theta \\ \sqrt{c-cx^2} \quad x=\sin\theta \quad \cdots \end{array}\right)$

$x=2\sin\theta$ $\overset{\#}{=}$ $\int x^2\cdot\sqrt{4-4\sin^2\theta}\cdot 2\cos\theta\cdot d\theta$

$\Downarrow$
$dx=2\cos\theta\, d\theta$

$dx = 2\cos\theta\, d\theta$

$$= \int x^2 \cdot 2\sqrt{\cos^2\theta} \cdot 2\cos\theta\, d\theta$$

$$= \int 4\sin^2\theta \cdot 2 \cdot \cos\theta \cdot 2\cos\theta\, d\theta$$

$$= 16 \int \sin^2\theta \cdot \cos^2\theta\, d\theta$$

$$= 4 \int (2\sin\theta \cdot \cos\theta)^2\, d\theta$$

$$= 4 \int (\sin 2\theta)^2\, d\theta$$

$$= 4 \int \frac{1-\cos 4\theta}{2}\, d\theta$$

$2\int 1\, d\theta$

$$= 4\int \frac{1}{2}\, d\theta - 4\int \frac{1}{2}\cos 4\theta\, d\theta$$

$$= 2\theta - 2\int \cos 4\theta\, d\theta$$

$$= 2\theta - \frac{1}{2}\sin 4\theta + C$$

(I). $\theta = \sin^{-1}\left(\frac{x}{2}\right)$

$$\boxed{2\cdot \sin^{-1}\left(\frac{x}{2}\right)} - \frac{1}{2}\sin\left(4\cdot \sin^{-1}\left(\frac{x}{2}\right)\right) + C.$$

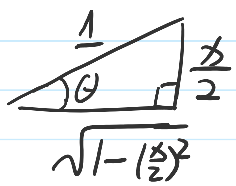
(II).

$$-\frac{1}{2}(2\sin 2\theta \cdot \cos 2\theta)$$

$$-\frac{1}{2}(2\cdot 2\cdot \sin\theta \cdot \cos\theta \cdot (\cos^2\theta - \sin^2\theta))$$

$\sin\theta = \frac{x}{2}$

$x = 2\sin\theta$



$\cos\theta = \sqrt{1-\left(\frac{x}{2}\right)^2}$

$$\Rightarrow 2\sin^{-1}\left(\frac{x}{2}\right) - \frac{1}{2}x\cdot\sqrt{4-x^2}\left(1-\frac{x^2}{2}\right) + C \ \#$$

(C). $\displaystyle\int \frac{3x^4 - 5x^3 + x^2 + 2x + 1}{3x^3 - 2x^2 - x}\, dx$

$(x-1)(3x^3-2x^2-x) + (x+1)$.

(1) Long division.

$$\begin{array}{r|l} & \quad x - 1 \\ \hline 3x^3-2x^2-x & 3x^4 - 5x^3 + x^2 + 2x + 1 \\ & 3x^4 - 2x^3 - x^2 \\ \hline & 0 \ -3x^3 + 2x^2 + 2x + 1 \end{array}$$

$$\begin{array}{r} 0 \quad -3x^3+2x^2+2x+1 \\ -3x^3+2x^2+x \\ \hline (x+1) \end{array}$$

(2) $$= \int (x-1) + \frac{x+1}{3x^3-2x^2-x}\, dx$$

$$= \frac{1}{2}x^2 - x + \int \frac{x+1}{3x^3-2x^2-x}\, dx$$

$$= \frac{1}{2}x^2 - x + \int \frac{x+1}{x(3x^2-2x-1)}\, dx$$

$$\frac{1}{x^2+1}$$

$$= \frac{1}{2}x^2 - x + \int \left(\frac{x+1}{x(3x+1)(x-1)}\right) dx$$

(3).

$$\frac{A}{x} + \frac{B}{3x+1} + \frac{C}{x-1}.$$

$$\frac{x+1}{\boxed{}} = \frac{A(3x+1)(x-1)+Bx(x-1)+Cx(3x+1)}{x(3x+1)(x-1)}$$

Pick $x=0$, $-A=1$, $\Rightarrow A=-1$.

Pick $x=1$, $4C=2$, $\Rightarrow C=\frac{1}{2}$

Pick $x=-\frac{1}{3}$, $\frac{4B}{9}=\frac{2}{3}$, $\Rightarrow B=\frac{3}{2}$.

(4) $$= I = \frac{1}{2}x^2 - x + \int \frac{-1}{x}\, dx + \int \frac{3}{2(3x+1)}\, dx + \int \frac{1}{2(x-1)}\, dx.$$

$$= \frac{1}{2}x^2 - x - \ln|x| + \frac{1}{2}\ln|3x+1| + \frac{1}{2}\ln|x-1| + C.$$

$\ln|x| + 2\ln|3x+1| + \underline{2\ln|x-1|} + C.$

(9) $\displaystyle\int \frac{x^2-5x-5}{(x-2)(x^2+2x+3)}\,dx$

(1) $\Rightarrow \dfrac{A}{x-2} + \dfrac{Bx+C}{x^2+2x+3}$

$\Rightarrow \dfrac{x^2-5x-5}{\boxed{}} = \dfrac{A(x^2+2x+3)+(Bx+C)(x-2)}{(x-2)(x^2+2x+3)}$

(2) Pick $x=2$, $11A = -11 \Rightarrow A = -1.$

$x^2-5x-5 = -x^2-2x-3 + Bx^2 - 2Bx + Cx - 2C.$

$2x^2-3x-2 = Bx^2 + (C-2B)x - 2C.$

$\begin{cases} \text{Compare } x^2 : B = 2. \\ \text{Compare } 1 : -2 = -2C \Rightarrow C = 1. \end{cases}$

(3) $\Rightarrow \displaystyle\int \frac{-1}{x-2}\,dx + \int \frac{2x+1}{x^2+2x+3}\,dx.$

$\dfrac{1}{(x^2+2x+3)'}dy$

$\Downarrow$

$\dfrac{1}{2x+2}dy$

$= -\ln|x-2| + \displaystyle\int \frac{2x+2-1}{x^2+2x+3}\,dx$

$= -\ln|x-2| + \displaystyle\int \frac{2x+2}{x^2+2x+3}\,dx - \int \frac{1}{x^2+2x+3}\,dx \rightarrow \int \frac{1}{1+x^2}dx$

$= -\ln|x-2| \overset{y=x^2+2x+3}{+} \displaystyle\int \frac{2x+2}{y} \cdot \frac{1}{\ldots}\,dy$ ... $= \tan^{-1}(x)$

$$= -\ln|x-2| + \underset{\substack{y = x^2+2x+3 \\ \Downarrow \\ dy = (2x+2)dx \\ \Downarrow \\ dx = \frac{1}{2x+2}dy}}{} \int \frac{\cancel{2x+2}}{y} \cdot \frac{1}{\cancel{2x+2}} dy - \int \frac{1}{(x+1)^2+2} dx \qquad = \tan^{-1}(x)$$

$$\int \frac{\frac{1}{2}}{\frac{(x+1)^2}{2}+1} dx$$

$$= -\ln|x-2| + \ln|x^2+2x+3| - \frac{1}{2}\int \frac{1}{\left(\frac{x+1}{\sqrt{2}}\right)^2+1} dx + C$$

$$= -\ln|x-2| + \ln|x^2+2x+3| - \frac{1}{2}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + C.$$

(k)(*) $\displaystyle\int \frac{6x^3-27x^2+5x-1}{(x-2)^2(4x^2+1)} dx.$

$$= \frac{A}{(x-2)} + \frac{B}{(x-2)^2} + \frac{Cx+D}{4x^2+1}.$$

(*)

$$= \frac{A(x-2)(4x^2+1) + B(4x^2+1) + (Cx+D)(x-2)^2}{(x-2)^2(4x^2+1)}$$

Pick $x = 2$, $17B = -51 \Rightarrow B = -3.$

$$6x^3-27x^2+5x-1 = -12x^2-3 + A(x-2)(4x^2+1) + (Cx+D)(x-2)^2$$

$$(6x^2-3x-1)\cancel{(x-2)} = A\cancel{(x-2)}(4x^2+1) + (Cx+D)(x-2)^{\cancel{2}}.$$

$$(6x^2-3x-1) = A(4x^2+1) + (Cx+D)(x-2)$$

$$(6x^2-3x-1) = A(4x^2+1)+(Cx+D)(x-2)$$

Pick $x=2$
$A=1$

Compare x^2: $6 = 4A+C$

Compare x: $-3 = (-2C+D)$

Compare 1: $-1 = A-2D$

$$\Rightarrow \begin{cases} A=1 \\ C=2 \\ D=1 \end{cases}$$

$$I = \int \frac{1}{x-2}dx - 3\int \frac{1}{(x-2)^2}dx + \int \frac{2x+1}{4x^2+1}dx$$

$\frac{1}{(4x^2+1)}dy$

$\frac{1}{8x}dy$

$$= \ln|x-2| - 3\int \frac{1}{(x-2)^2}d(x-2) + \int \frac{2x}{4x^2+1}dx + \int \frac{1}{4x^2+1}dx$$

$$= \ln|x-2| + \frac{3}{x-2} + \frac{1}{4}\int \frac{1}{4x^2+1}d(4x^2+1) + \frac{1}{2}\int \frac{1}{(2x)^2+1}d(2x)$$

$8x\,dx$

$$= \ln|x-2| + \frac{3}{x-2} + \frac{1}{4}\ln|4x^2+1| + \frac{1}{2}\tan^{-1}(2x). \quad \#$$

(III). Reduction formula.

P9: $I_n = \int_1^e x^a(\ln x)^n dx$

$\int x^a \cdot \sin x\, dx$

$\int x^a \cdot \ln x\, dx$

(a) $\boxed{I_n = \frac{e^{a+1}}{a+1} - \frac{n}{a+1}I_{n-1}.}$

P12

$I_n = \int \cos^n x\, dx$

$I_n \sim I$

$I_n = \int_1^e (\ln x)^n d\left(\frac{1}{a+1}x^{a+1}\right)$, $\boxed{a=-1}$

$$I_n = \int_1 (\ln x) \, d(\underbrace{\tfrac{1}{a+1} x}_{dv}) , \quad \boxed{a = -1}$$

$u = \ln x$

$I_n \sim I_{n-2}$

$$= (\ln x)^n \cdot \frac{1}{a+1} x^{a+1} \Big|_1^{e} - \int_1^e \frac{1}{a+1} x^{a+1} \, d((\ln x)^n)$$

$x^{-1} = d(\ln x)$

$$= 1^n \cdot \frac{1}{a+1} \cdot e^{a+1} - 0 - \int_1^e \frac{1}{a+1} x^{a+1} \cdot n \cdot (\ln x)^{n-1} \cdot (\ln x)' \, dx$$

$$= \frac{e^{a+1}}{a+1} - \int_1^e \frac{1}{a+1} x^{a\cancel{+1}} \cdot n \cdot (\ln x)^{n-1} \cdot \frac{1}{\cancel{x}} \, dx$$

$$= \frac{e^{a+1}}{a+1} - \frac{n}{a+1} \underbrace{\int_1^e x^a \cdot (\ln x)^{n-1} \, dx}_{I_{n-1}}$$

$$I_n = \frac{e^{a+1}}{a+1} - \frac{n}{a+1} \cdot I_{n-1}. \checkmark$$

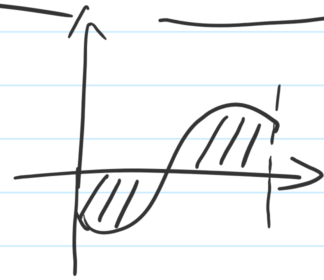
$I_3 = \dots \quad I_2$

Chapter 3

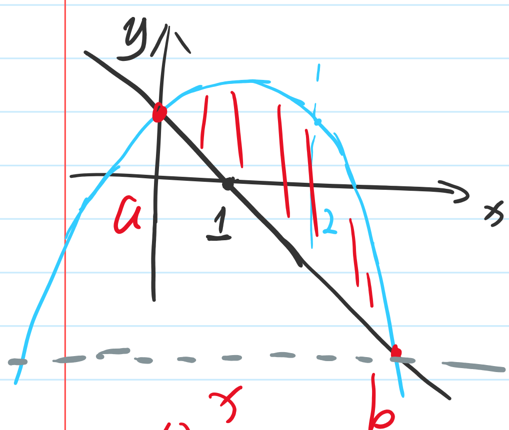
Saturday, March 16, 2019 5:07 PM

(I) Area of the region: $\int_a^b f(x)\,dx = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(\xi_i)\Delta x_i.$

$\Rightarrow \int_a^b |f(x)|\,dx$

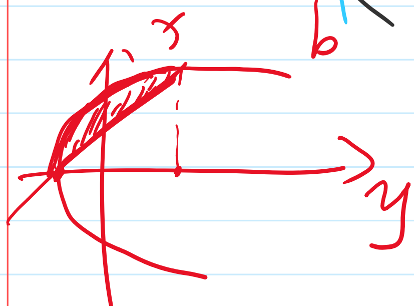


P1. (d) Area of region bounded by $\begin{cases} y = -x^2+2x+1. \\ x+y=1. \end{cases}$



$\text{Area} = \int_a^b \underline{(-x^2+2x+1)} - \underline{(-x+1)}\,dx$

$a, b \Rightarrow \begin{cases} y=-x^2+2x+1 \\ x+y=1 \end{cases} \Rightarrow \begin{cases} x_1=0 \\ y_1=1 \end{cases} \text{ or } \begin{cases} x_2=3 \\ y_2=-2. \end{cases}$

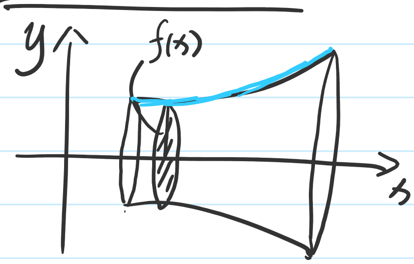


$= \int_0^3 (-x^2+3x)\,dx$

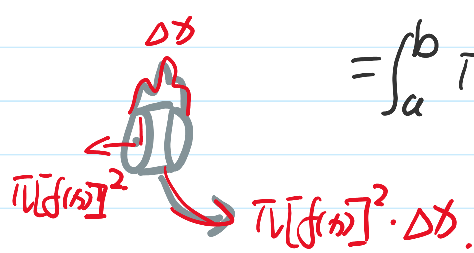
$= -\frac{x^3}{3} + \frac{3x^2}{2}\Big|_0^3 = \frac{9}{2}.$

(II) Volume:

$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(\xi_i)\Delta x_i = \int_a^b A(x)\,dx$



$= \int_a^b \pi[f(x)]^2\,dx$



P4. (d)

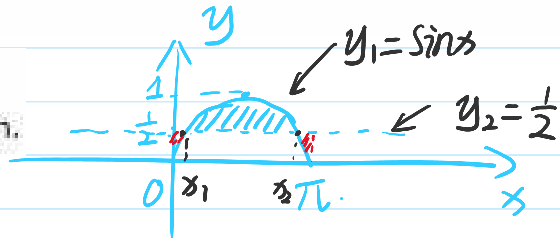
(d) Find the volume of the solid generated by rotating the region above $y = \frac{1}{2}$ and below $y = \sin x$ for $0 \le x \le \pi$ about

(d) Find the volume of the solid generated by rotating the region above $y=\frac{1}{2}$ and below $y=\sin x$ for $0\le x\le\pi$ about

(i) the x-axis for 1 complete revolution.

(ii) the y-axis for 1 complete revolution.

(iii) the line $y=\frac{1}{2}$ for 1 complete revolution.



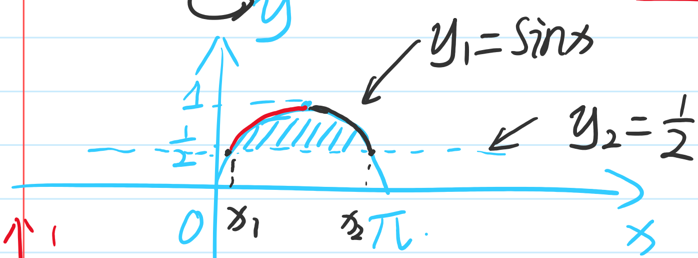
(i) $\begin{cases} y=\sin x \\ y=\frac{1}{2}\end{cases} \Rightarrow \sin x=\frac{1}{2} \Rightarrow \begin{cases} x_1=\frac{\pi}{6} \\ y_1=\frac{1}{2}\end{cases}$ or $\begin{cases} x_2=\frac{5\pi}{6} \\ y_2=\frac{1}{2}.\end{cases}$

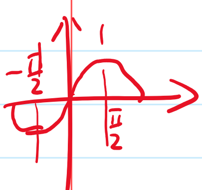
$$\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}\pi[f_1(x)]^2-\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}\pi[f_2(x)]^2\,dx$$

$$=\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}\left[\pi(\sin x)^2-\pi\left(\tfrac{1}{2}\right)^2\right]dx$$

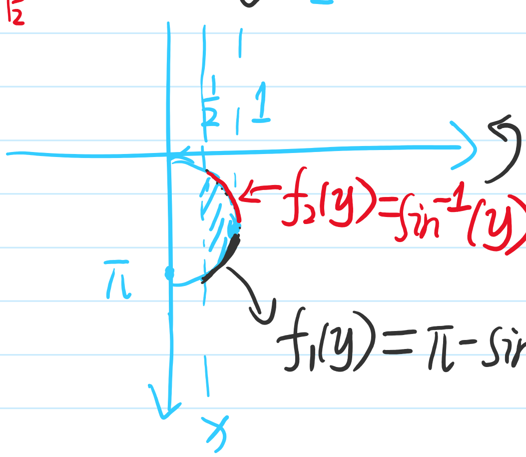
$$=\pi\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}\frac{1-\cos 2x}{2}\,dx-\pi\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}\frac{1}{4}\,dx=\frac{\pi^2}{6}+\frac{\sqrt{3}\pi}{4}.$$

(ii) $x=\sin^{-1}(y)\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ Method (I): Disk method





$\Downarrow$ 2



$$V=\int_{\frac{1}{2}}^{1}\pi[f_1(y)]^2dy-\int_{\frac{1}{2}}^{1}\pi[f_2(y)]^2dy$$

$$=\int_{\frac{1}{2}}^{1}\pi[\pi-\sin^{-1}(y)]^2dy-\int_{\frac{1}{2}}^{1}\pi[\sin^{-1}(y)]^2dy$$

$$=\pi\int_{\frac{1}{2}}^{1}(\pi^2-2\pi\sin^{-1}(y))\,dy.$$

$$=\pi^3\int_{\frac{1}{2}}^{1}dy-2\pi^2\int_{\frac{1}{2}}^{1}\sin^{-1}y\,dy$$

$$=\pi^3\,y\Big|_{\frac{1}{2}}^{1}-2\pi^2\left(\sin^{-1}y\cdot y\Big|_{\frac{1}{2}}^{1}-\int_{\frac{1}{2}}^{1}y\,d(\sin^{-1}y)\right)$$

$$=\frac{1}{2}\pi^3-2\pi^2\left(\frac{\pi}{2}-\frac{\pi}{12}-\int_{\frac{1}{2}}^{1}\frac{y}{\sqrt{1-y^2}}\,dy\right)$$

$$=\frac{1}{2}\pi^3-2\pi^2\left(\frac{\pi}{2}-\frac{\pi}{12}-\int_{\frac{1}{2}}^{1}\frac{y}{\sqrt{1-y^2}}\,dy\right)$$

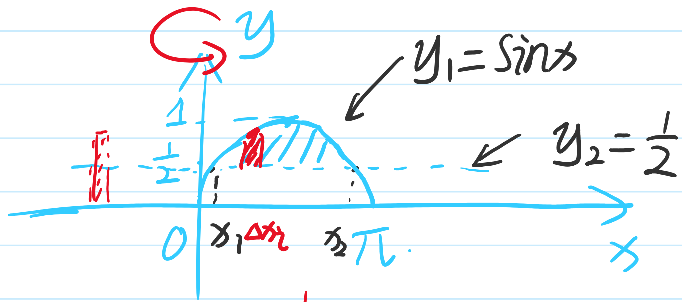
$$=\frac{\pi^3}{2}-\frac{5\pi^3}{6}-\pi^2\int_{\frac{3}{4}}^{0}\frac{1}{\sqrt{z}}\,dz \qquad \Downarrow z=1-y^2 \Rightarrow dz=-2y\,dy \Rightarrow dy=-\frac{1}{2y}dz$$

$$=-\frac{\pi^3}{3}+\sqrt{3}\pi^2.$$

Method (II): shell method:

$$V_y=\lim_{n\to\infty}\sum_{i=1}^{n}2\pi x_i f(x_i)\,\Delta x_i$$

$$=\int_a^b 2\pi x\cdot f(x)\,dx$$



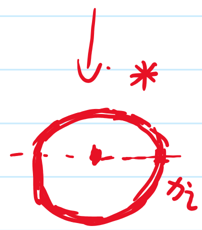
Here,

$$V_y=\int_a^b 2\pi x\cdot(f_1-f_2)\,dx$$

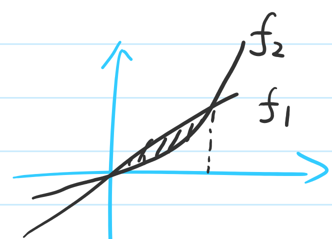
$$=2\pi\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}x\left(\sin x-\frac{1}{2}\right)dx$$

$$=2\pi\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}x\cdot\sin x\,dx-\pi\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}x\,dx$$

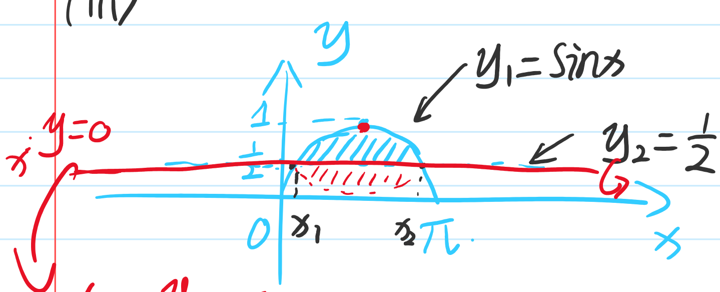
$$=\sqrt{3}\pi^2-\frac{\pi^3}{3}.$$



$$\underbrace{2\pi x_i}_{\text{length}}\cdot\underbrace{f(x_i)}_{\text{Height}}\cdot\underbrace{\Delta x_i}_{\text{thickness}}$$



(iii)



x : $y_1=\sin x-\frac{1}{2}$.

Rotate about $y=\frac{1}{2}$.

$$V=\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}\pi\left(\sin x-\frac{1}{2}\right)^2dx$$

$$=\pi\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}\sin^2x\,dx-\pi\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}\sin x\,dx+\frac{\pi}{4}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}1\,dx$$

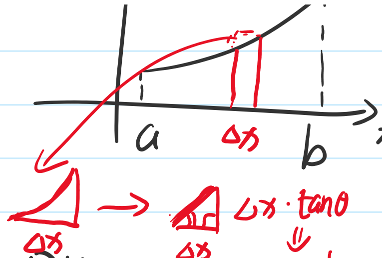
$$=\frac{\pi^2}{2}-\frac{3\sqrt{3}}{4}\pi$$

(III) Arc length:

① $y=f(x)$

$$l=\int_a^b\sqrt{1+[f'(x)]^2}\,dx.$$

Arc length:

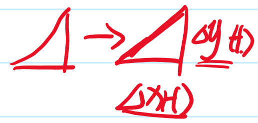


$l = \int_a^b \sqrt{1+[f'(x)]^2}\,dx.$

$\Delta x \cdot f'(x)$

$\sqrt{(\Delta x)^2+[\Delta x \cdot f'(x)]^2} \Rightarrow \sqrt{1+[f'(x)]^2}\,\Delta x$

② $\begin{cases} x(t) \\ y(t) \end{cases}$



$l = \int_0^t \sqrt{[x'(t)]^2+[y'(t)]^2}\,dt.$ $\quad (\frac{dx}{dt})^2$

Assignment 2 : P4

Length of the curve $x = \frac{y^3}{12} + \frac{1}{y}$, $1 \le y \le 2$.

$l = \int_1^2 \sqrt{1+[x'(y)]^2}\,dy \qquad x'(y) = \frac{1}{4}y^2 - \frac{1}{y^2}$

$= \int_1^2 \sqrt{1+(\frac{1}{4}y^2 - \frac{1}{y^2})^2}\,dy$

$= \int_1^2 \sqrt{1+\frac{1}{16}y^4 - \frac{1}{2} + \frac{1}{y^4}}\,dy$

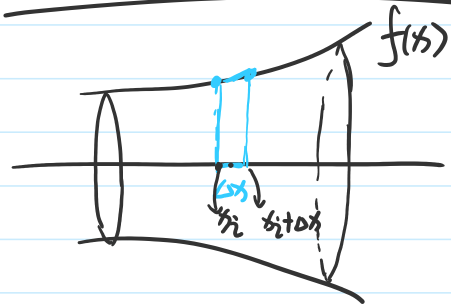
$= \int_1^2 \sqrt{(\frac{1}{4}y^2 + \frac{1}{y^2})^2}\,dy$

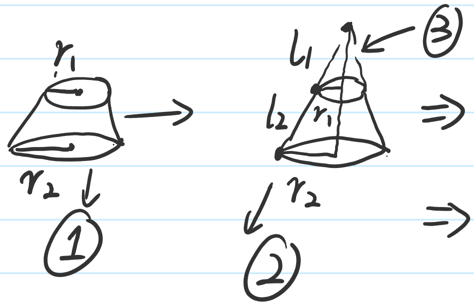
$= \int_1^2 (\frac{1}{4}y^2 + \frac{1}{y^2})\,dy$

$= (\frac{1}{4}\cdot\frac{1}{3}y^3 - \frac{1}{y})\Big|_1^2$

$= \frac{13}{12}.$ #

(IV) Surface Area





$\Rightarrow \frac{r_1}{r_2} = \frac{l_1}{l_1+l_2}$

$\Rightarrow l_2 = \frac{r_1 l_1}{r_2 - r_1}$

$S① = S② - S③$

$= \pi r_2(l_1+l_2) - \pi r_1 l_1$

$= \pi l_2(r_1+r_2)$

$\therefore S(\Delta x) = \pi \cdot \sqrt{1+[f'(x_i)]^2}\,\Delta x \cdot (f(x_i) + f(x_i+\Delta x))$

$= \pi \cdot \sqrt{1+[f'(x_i)]^2} \cdot \Delta x \cdot 2f(x_i)$

$= 2\pi f(x_i) \cdot \sqrt{1+[f'(x_i)]^2}\,\Delta x.$

$S = \lim_{n\to\infty} \sum_{i=1}^{n} S(\Delta x) = \lim \sum_{i=1}^{n} 2\pi f(x_i) \cdot \sqrt{1+[f'(x_i)]^2} \cdot \Delta x$

$$S=\lim_{n\to\infty}\sum_{i=1}^{n}S(\Delta x)=\lim_{n\to\infty}\sum_{i=1}^{n}2\pi f(x_i)\cdot\sqrt{1+[f'(x_i)]^2}\cdot\Delta x$$

$$=2\pi\int_a^b f(x)\cdot\sqrt{1+[f'(x)]^2}\,dx.$$

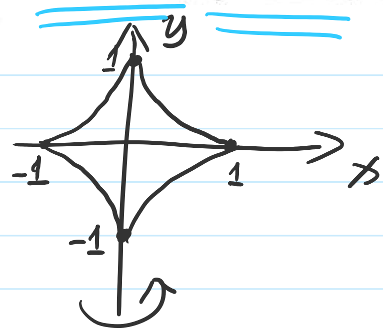
(Remark: works for $f(x)$ lies above x-axis.

or $\begin{cases} x=x(t) \\ y=y(t) \end{cases}$ $a\le t\le b.$ $\Rightarrow S=2\pi\int_a^b y(t)\sqrt{[x'(t)]^2+[y'(t)]^2}\,dt$

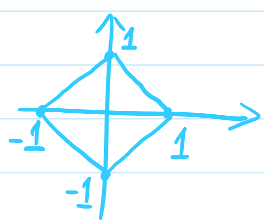
Mid-term 2 − 2017: P1(b)

(b) Find the surface area of the solid by revolving the Astroid: $x=\cos^3 t$, $y=\sin^3 t$, $0\le t\le 2\pi$, about the y-axis. [18]

$\cos^2 t+\sin^2 t=1 \Rightarrow x^{\frac{2}{3}}+y^{\frac{2}{3}}=1.$



$|x|^1+|y|^1=1.$

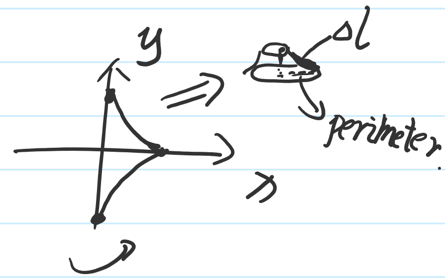


when $0\le t\le 2\pi$, $\Rightarrow -1\le\cos t\le 1 \Rightarrow -1\le x\le 1.$

$\Rightarrow -1\le\sin t\le 1\Rightarrow -1\le y\le 1.$

$\frac{dx}{dt}=3\cos^2 t(-\sin t)$, $\frac{dy}{dt}=3\sin^2 t\cdot\cos t$

$$S=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\underbrace{2\pi\,x(t)}_{\text{perimeter}}\cdot\underbrace{\sqrt{[x'(t)]^2+[y'(t)]^2}\,dt}_{\Delta l}$$



$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}2\pi\cos^3 t\cdot\sqrt{9\cos^4 t\sin^2 t+9\sin^4 t\cdot\cos^2 t}\,dt.$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}2\pi\cos^3 t\sqrt{9\cos^2 t\sin^2 t(\cos^2 t+\sin^2 t)}\,dt.$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\pi\cos^3 t\ \sqrt{9\cos^2 t\sin^2 t\,(\underbrace{\cos^2 t+\sin^2 t}_{1})}\,dt.$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\pi\cos^3 t\cdot|3\cos t\cdot\sin t|\,dt.$$

$$=2\int_{0}^{\frac{\pi}{2}} 6\pi\cos^4 t\cdot\sin t\,dt$$

$$\Rightarrow 2\int_{0}^{\frac{\pi}{2}} 6\pi\cos^4 t\,d(-\cos t)\overset{\cos t=z}{\Longrightarrow}12\pi\int_{1}^{0} z^4\,d(-z)\Rightarrow\frac{12\pi}{5}\qquad\#$$