Chapter 1

Sunday, January 19, 2020 8:49 PM

MA0101 Tutorial Class (TB3) session.

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Chapter 1. Vector : Force. { direction : ↑ ↗ ; magnitude : → quantity.

Scalar : Number / value.

"+" → "×" $|\cdot|$ $|-2| = 2$ $|2| = 2$

Vector → "+" "×"

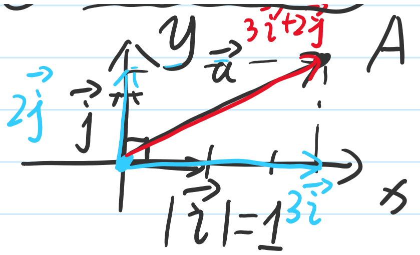
Representation of vector :

① $\vec{a}$ { "direction" : ✓ unit vector : $\dfrac{\vec{a}}{|\vec{a}|}$ → $\left|\dfrac{\vec{a}}{|\vec{a}|}\right| = 1$.

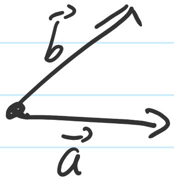
magnitude : ✓ $|\vec{a}|$

$$|\vec{a}| \times \frac{\vec{a}}{|\vec{a}|}$$

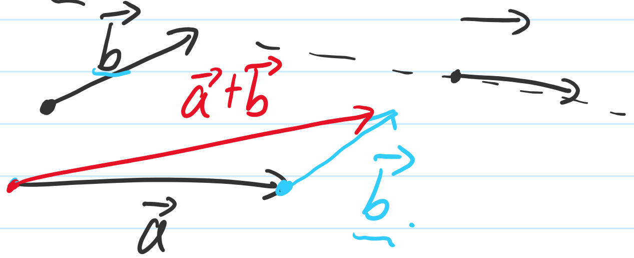
② Fundamental vectors : $\underset{\triangle}{\vec{i}}$, $\underset{\triangle}{\vec{j}}$ 2D , $\underset{\triangle}{\vec{k}}$

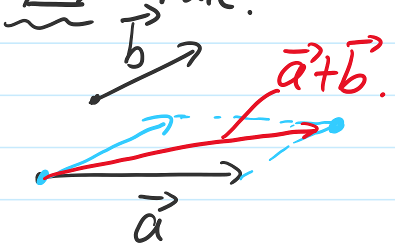


$$\vec{a} = 3\vec{i} + 2\vec{j}$$

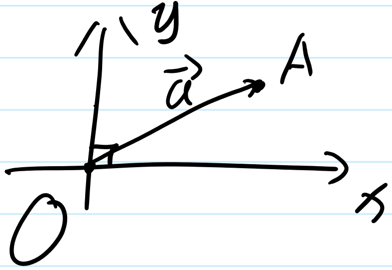


"+" { (a) tip-to-tail rule.
(b) ▱ rule.





③ Cooridinate system:



$$\vec{a} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} = (3,2).$$

$\Downarrow$

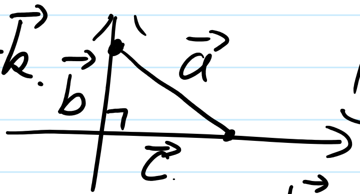
$$\vec{OA} = (3,2)$$

"3D"

P1: let $\vec{a} = 2\vec{i} + (-2)\vec{j} + \vec{k}$, $\vec{b} = \vec{i} + 8\vec{j} - 4\vec{k}$, $\vec{C} = 12\vec{i} - 4\vec{j} - 3\vec{k}$.

(a). $|\vec{a}| = \sqrt{2^2 + (-2)^2 + (1)^2} = \sqrt{9} = 3.$ ✓ Pythagorean. theorom

$$\frac{\vec{a}}{|\vec{a}|} = \frac{2\vec{i} - 2\vec{j} + \vec{k}}{3} = \frac{2}{3}\vec{i} - \frac{2}{3}\vec{j} + \frac{1}{3}\vec{k}.$$



$$|\vec{a}|^2 = |\vec{b}|^2 + |\vec{c}|^2$$

$$|\vec{a}| = \sqrt{|\vec{b}|^2 + |\vec{c}|^2}.$$

✓ $$\left|\frac{\vec{a}}{|\vec{a}|}\right| = 1 = \sqrt{\left(\frac{2}{3}\right)^2 + \left(-\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2}$$

✓ $$|\vec{a} + \vec{b}| = |(2\vec{i} - 2\vec{j} + \vec{k}) + (\vec{i} + 8\vec{j} - 4\vec{k})|$$

✓ $|\underline{a+b}| = |(2\vec{i}-2\vec{j}+\vec{k})+(\vec{i}+8\vec{j}-4\vec{k})|$.

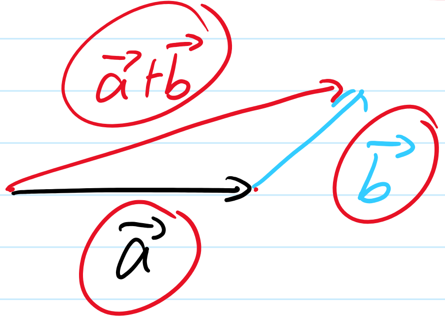
$= |3\vec{i}+6\vec{j}-3\vec{k}|$.

$= \sqrt{(3)^2+(6)^2+(-3)^2} = \boxed{\sqrt{54}} = 3\sqrt{6}$.

✓ $|\vec{a}|+|\vec{b}| = \sqrt{2^2+(-2)^2+1^2} + \sqrt{1^2+8^2+(-4)^2} = \boxed{12}$.

$$\boxed{\text{"}\ |\vec{a}+\vec{b}| \leq |\vec{a}|+|\vec{b}|\ \text{"}}$$

Triangular inequality.



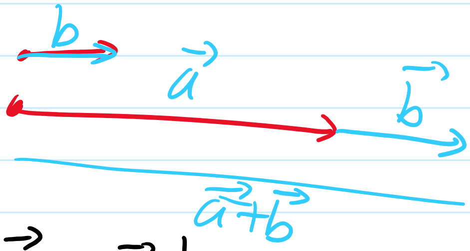
$3-2 = 3+(-2)$

$\vec{a}-\vec{b} = \vec{a}+(-\vec{b})$

$\vec{b}$

$-\vec{b}$

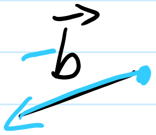
$|\vec{a}|+|\vec{b}| \geq |\vec{a}+\vec{b}|$



"−" $\vec{a}-\vec{b} = \vec{a}+(-\vec{b})$

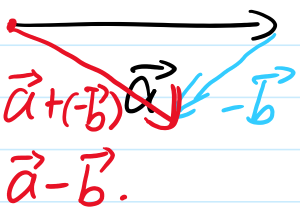
$|\vec{a}-\vec{b}| = |(2\vec{i}-2\vec{j}+\vec{k})-(\vec{i}+8\vec{j}-4\vec{k})|$

$= |\vec{i}-10\vec{j}+5\vec{k}| = \boxed{\sqrt{126}}$.



$|\vec{a}|-|\vec{b}| = \sqrt{2^2+(-2)^2+1^2} - \sqrt{1^2+8^2+(-4)^2}$

$= \boxed{-6}$.



$= \vec{a}-\vec{b}$.

$|\vec{a}-\vec{b}| \geq ||\vec{a}|-|\vec{b}||$

$$|\vec{a}-\vec{b}| \geq \left||\vec{a}|-|\vec{b}|\right|$$

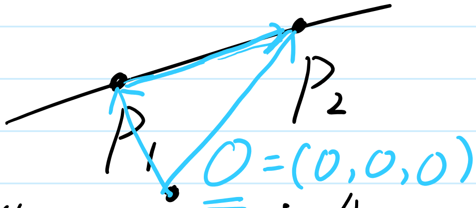
multiplication:

① scalar multiplication: $\lambda \vec{a}$ → vector

② dot/inner multiplication: $\vec{a} \cdot \vec{b}$ → scalar } scalar product

③ cross multiplication: $\vec{a} \times \vec{b}$ → vector } vector product

P2: let l be straight line through $P_1=(1,2,2)$, $P_2=(0,2,5)$



(a) "unit vector" in direction of P_1P_2

$\vec{P_1P_2} = \vec{OP_2} - \vec{OP_1}$ $=(-1,0,3)$

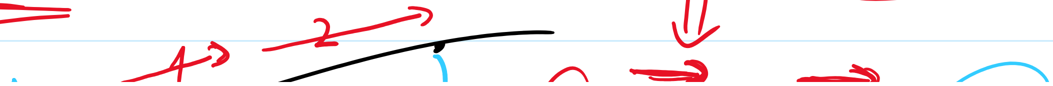
$=(0,2,5)-(1,2,2)$

$=(-1,0,3)$

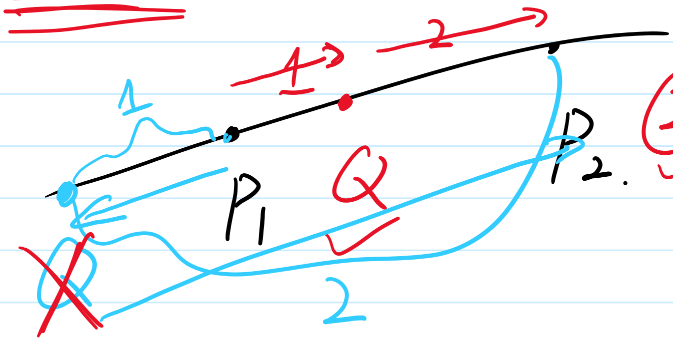
$$\widehat{P_1P_2} = \frac{\vec{P_1P_2}}{|\vec{P_1P_2}|} = \frac{-\vec{i}+3\vec{k}}{\sqrt{(-1)^2+0^2+3^2}} = -\frac{1}{\sqrt{10}}\vec{i}+\frac{3}{\sqrt{10}}\vec{k}.$$

(b). Point Q on l, such that Q is on the ~~line~~ P_1P_2 and $\boxed{P_1Q : QP_2 = 1:2}$.

① segment.



① $\Downarrow$



② $\overrightarrow{P_1Q} : \overrightarrow{QP_2} = 1:2 = \frac{1}{2}$

$= \ominus\frac{1}{2}$

$\boxed{Q : (x, y, z)}$

$\overrightarrow{P_1Q} = \frac{1}{3}\overrightarrow{P_1P_2} = \frac{1}{3}(-1,0,3) = (-\frac{1}{3},0,1)$

$\Downarrow$

$(x-1, y-2, z-2) \Longrightarrow \begin{cases} x = \frac{2}{3} \\ y = 2 \\ z = 3 \end{cases}$. ✓

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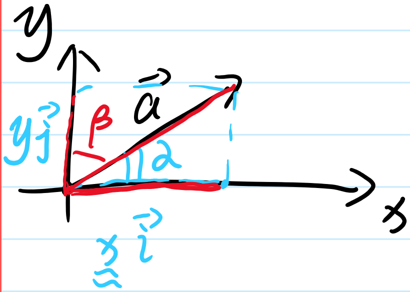
① unit vector $\Longleftrightarrow$ Direction cosines

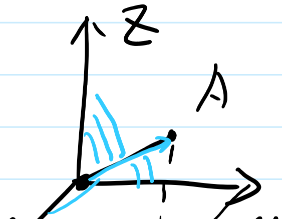


$\vec{a} \longrightarrow \begin{cases} \text{direction} \\ \text{magnitude.} \end{cases}$ $\vec{a} = \begin{pmatrix} x \\ y \end{pmatrix}$

" $\begin{cases} \cos\alpha = \frac{x}{|\vec{a}|} = \frac{x}{\sqrt{x^2+y^2}}, \\ \cos\beta = \frac{y}{|\vec{a}|} = \frac{y}{\sqrt{x^2+y^2}}, \end{cases}$ "

$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{x\vec{i}+y\vec{j}}{\sqrt{x^2+y^2}} = \underbrace{\frac{x}{\sqrt{x^2+y^2}}}_{\cos\alpha}\vec{i} + \underbrace{\frac{y}{\sqrt{x^2+y^2}}}_{\cos\beta}\vec{j}. = \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} \\ \frac{y}{\sqrt{x^2+y^2}} \end{pmatrix} = \begin{pmatrix} \cos\alpha \\ \cos\beta \end{pmatrix}$

unit vector.



$\overrightarrow{OA}$

$\hat{}$ $\begin{pmatrix} \cos\alpha \end{pmatrix}$ $\begin{pmatrix} \frac{x}{\sqrt{x^2+y^2+z^2}} \end{pmatrix}$

OA

$$\hat{OA} = \begin{pmatrix}\cos\alpha\\ \cos\beta\\ \cos\gamma\end{pmatrix} = \begin{pmatrix}\frac{x}{\sqrt{x^2+y^2+z^2}}\\ \frac{y}{\sqrt{x^2+y^2+z^2}}\\ \frac{z}{\sqrt{x^2+y^2+z^2}}\end{pmatrix}.$$

$= \begin{pmatrix}\cos\alpha\\ \cos\beta\end{pmatrix}.$

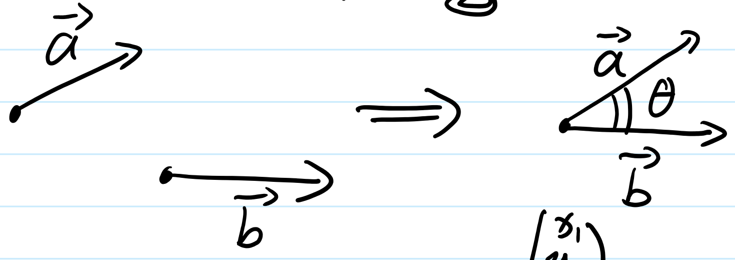
(2) "•" scalar product. inner product. dot product.

$\vec{a}\cdot\vec{b}$

"×" vector, outer product. cross product.

$\vec{a}\times\vec{b} =$

$$\vec{a}\cdot\vec{b} = |\vec{a}|\cdot|\vec{b}|\cdot\cos\theta$$ ✓



$$\begin{cases}\vec{a} = x_1\vec{i} + y_1\vec{j} + z_1\vec{k} \Rightarrow \begin{pmatrix}x_1\\ y_1\\ z_1\end{pmatrix}\\ \vec{b} = x_2\vec{i} + y_2\vec{j} + z_2\vec{k} \Downarrow \begin{pmatrix}x_2\\ y_2\\ z_2\end{pmatrix}.\end{cases} \Longrightarrow \vec{a}\cdot\vec{b} = x_1x_2 + y_1y_2 + z_1z_2.$$

"θ"

$\vec{a} = \underset{\triangle}{2}\vec{i} - \underset{\square}{2}\vec{j} + \underset{\square}{\vec{k}}$, $\vec{b} = \underset{\triangle}{\vec{i}} + \underset{\square}{8}\vec{j} - \underset{\square}{4}\vec{k}$.

(b) $\vec{a}\cdot\vec{b} = 2\times1 + (-2)\times8 + 1\times(-4) = 2 + (-16) + (-4)$

$= -18.$ ~~"22"~~.

(c) angle between $\vec{a}$ and $\vec{b}$.

"θ" $\cos\theta$.

$\vec{a}\cdot\vec{b}$ -18

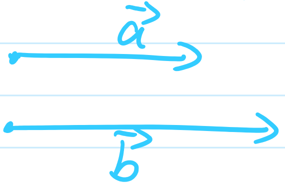
θ $\cos\theta$.

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|} = \frac{-18}{\sqrt{2^2+(-2)^2+1^2}\cdot\sqrt{1^2+8^2+(-4)^2}} = \frac{-18}{3\cdot 9} = -\frac{2}{3}.$$

$\Rightarrow \theta = \cos^{-1}(-\frac{2}{3}) \approx ?$

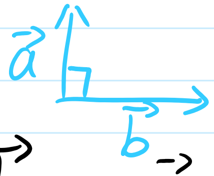
① $\theta = 0 \Rightarrow \cos\theta = 1 \Leftrightarrow \vec{a}\cdot\vec{b} = |\vec{a}|\cdot|\vec{b}|\cdot 1 = |\vec{a}|\cdot|\vec{b}|$

$\theta = 180^\circ \underset{=\pi}{} \Rightarrow \cos\theta = -1.$



② $\theta = 90^\circ = \frac{\pi}{2} \Rightarrow \cos\theta = 0 \Leftrightarrow \vec{a}\bullet\vec{b} = |\vec{a}|\cdot|\vec{b}|\cdot 0 = 0$

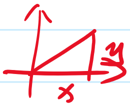
perpendicular.



P3: $\vec{a} = \vec{i} + 3\vec{j} - 2\vec{k}$, $\vec{b} = -2\vec{i} + \vec{j} + 3k$. ✓

(a) $\vec{a}\cdot\vec{b} = 1\times(-2) + 3\times 1 + (-2)\times 3 = -5.$

(b) angle between $\vec{a}$ and $\vec{b}$.

(i)  $\sqrt{x^2+y^2}$

$\vec{a} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|} = \frac{-5}{\sqrt{1^2+3^2+(-2)^2}\cdot\sqrt{(-2)^2+1^2+3^2}} = \frac{-5}{\sqrt{14}\cdot\sqrt{14}} = -\frac{5}{14}.$$

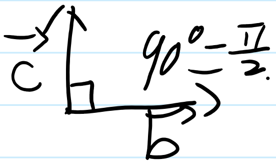
$\Rightarrow \theta = \cos^{-1}(-\frac{5}{14}) \approx 110.92^\circ.$

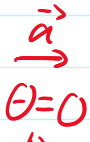
(ii) $|\vec{a}| = \sqrt{x^2+y^2+z^2}$

(c) $\vec{c} = 3\vec{i} + x\vec{j} - 2\vec{k}$ ✓, which is perpendicular to $\vec{b}$ ✓

$\vec{b}\cdot\vec{c} = \boxed{(-2)\times 3 + 1(x) + 3\times(-2) = \text{"}0\text{"}}$

$\Rightarrow x = 12$





$\vec{a}\cdot\vec{a} = |\vec{a}|\cdot|\vec{a}|\cdot\underset{1}{\underset{\parallel}{\cos\theta}}$

$= |\vec{a}|^2$

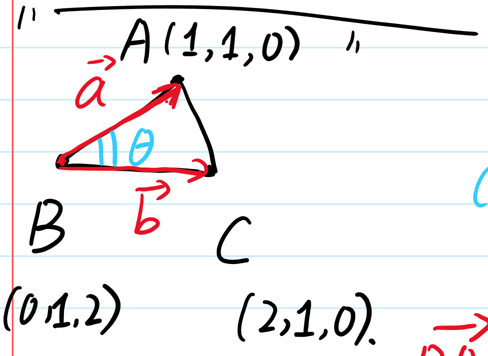
P4: Let $A = (1, 1, 0)$, $B = (0, 1, 2) \rightarrow \ldots$

$= |a|$ 1

P4: let $A=(1,1,0), B=(0,1,2)$ $\Rightarrow$ $|\vec{a}| = \sqrt{\vec{a}\cdot\vec{a}}$

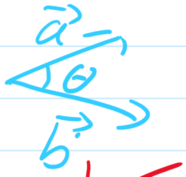
$C=(2,1,0)$.

$= \sqrt{x^2+y^2+z^2}$.



find "$\angle ABC$" $= \theta$

"$\cos\theta$"



$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|} = \frac{\vec{BA}\cdot\vec{BC}}{|\vec{BA}|\cdot|\vec{BC}|}$ ✓

$O:(0,0,0)$

$\vec{BA} = (1,0,-2)$, $\vec{BC} = (2,0,-2)$

$\vec{OA} - \vec{OB}$

$= (1,1,0) - (0,1,2)$

$= (1,0,-2)$

$\vec{OC} - \vec{OB}$.

$= (2,1,0) - (0,1,2)$

$= (2,0,-2)$.

$$\cos\theta = \frac{1\times 2 + 0\times 0 + (-2)\times(-2)}{\sqrt{1^2+0^2+(-2)^2}\cdot\sqrt{2^2+0^2+(-2)^2}} = \frac{6}{\sqrt{5}\cdot\sqrt{8}} = \frac{3}{\sqrt{10}}.$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{3}{\sqrt{10}}\right) \approx 18.43°.$$

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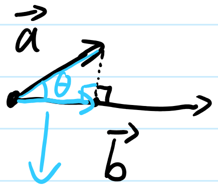
① Projection vector: $\text{Proj}_{\vec{b}}\vec{a}$



$|\text{Proj}_{\vec{b}}\vec{a}|$

dot

magnitude: $|\vec{a}|\cdot\cos\theta = \vec{a}\cdot\hat{b}$

direction: $\hat{b}$

$= |\vec{a}|\cdot|\hat{b}|\cdot\cos\theta$

$|\hat{b}| = 1$

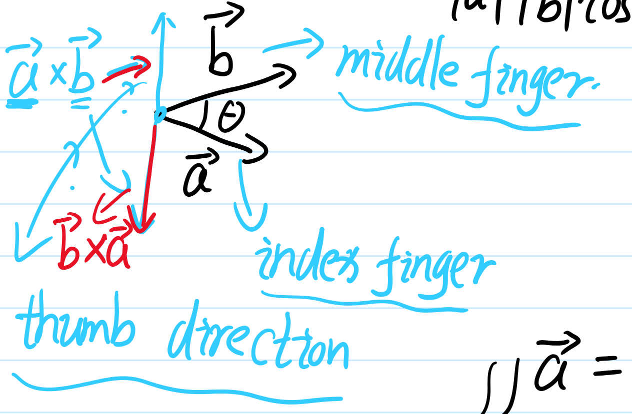
② Cross product / vector product / outer product.

② Cross product / vector product / outer product.

$\vec{a} \times \vec{b} :=$ vector $\Rightarrow$ magnitude: $|\vec{a}\times\vec{b}| = |\vec{a}|\cdot|\vec{b}|\cdot\sin\theta$

direction: right-hand rule.

$\vec{a} \cdot \vec{b} :=$ scalar $= |\vec{a}|\cdot|\vec{b}|\cdot\cos\theta$.



$\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$

$\boxed{\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}}$

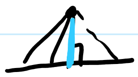
$\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$

$\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$.

$$``\vec{a}\times\vec{b}" = \det\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$

Summary: divide all Questions into 2 types

1st type:

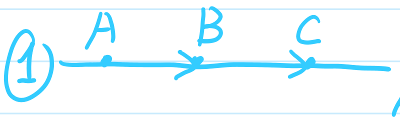
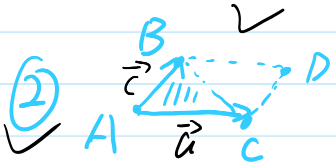
shortest distance		
point and line	point and plane	× ?line and line? (plane and plane
		
"·"	"·" and "×"	

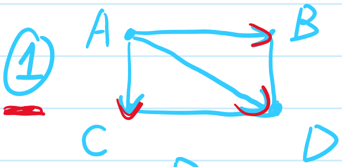
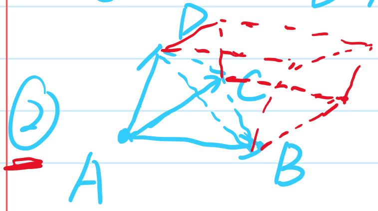
solve △

P4 P6

2nd type "dichotomy" ✓

dichotomy ✓

3 points = 2 vectors		collinear	area of triangle/parallelogram
① 	$\vec{AB}$ $\vec{AC}$	✓	0
② 	$\vec{AB}$ $\vec{AC}$	×	$\triangle ABC = \frac{1}{2}\lvert\vec{AB}\times\vec{AC}\rvert$ $\square ABCD = \lvert\vec{AB}\times\vec{AC}\rvert$

4 points = 3 vectors		Coplanar	volume of tetrahedron/parallelepipe
① 	$\vec{AB}$ $\vec{AC}$ $\vec{AD}$	✓	0
② 		×	$\frac{1}{6}\vec{AB}\cdot(\vec{AC}\times\vec{AD})$ $\vec{AB}\cdot(\vec{AC}\times\vec{AD})$

P1: (d) $\overset{3D}{\vec{a}}\times\overset{3D}{\vec{b}}$ $\quad\vec{a}=2\vec{i}-2\vec{j}+\vec{k},\ \vec{b}=\vec{i}+8\vec{j}-4\vec{k}.$

$$=\begin{pmatrix}2\\-2\\1\end{pmatrix}\times\begin{pmatrix}1\\8\\-4\end{pmatrix}$$

$$=\det\begin{pmatrix}\vec{i}&\vec{j}&\vec{k}\\2&-2&1\\1&8&-4\end{pmatrix}_{3\times3}=\overset{+}{\cancel{(-1)}}^{1+1}\det\begin{pmatrix}-2&1\\8&-4\end{pmatrix}_{2\times2}\vec{i}$$

1st row 1st column

$$\overset{-}{}+\cancel{(-1)}^{1+2}\det\begin{pmatrix}2&1\\1&-4\end{pmatrix}\vec{j}$$

1st row 2nd column

$$+\overset{+}{\cancel{(-1)}}^{1+3}\det\begin{pmatrix}2&-2\\1&8\end{pmatrix}\vec{k}$$

1st row 3rd column

2x2 matrix

$$\det\begin{pmatrix}a_1&a_2\\a_3&a_4\end{pmatrix}=a_1a_4-a_2a_3$$

$$=0\vec{i}-(-9)\vec{j}+18\vec{k}$$

$$= 0\vec{i} - (-9)\vec{j} + 18\vec{k}$$

$\vec{c} \times \vec{a}$ $$= 9\vec{j} + 18\vec{k} = \begin{pmatrix} 0 \\ 9 \\ 18 \end{pmatrix}.$$

$\vec{a} \times (\vec{b} \times \vec{c})$ $\qquad \vec{c} = 12\vec{i} - 4\vec{j} - 3\vec{k}$

1^{st} step: $\vec{b} \times \vec{c} = \begin{pmatrix} 1 \\ 8 \\ -4 \end{pmatrix} \times \begin{pmatrix} 12 \\ -4 \\ -3 \end{pmatrix}$

$$= \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 8 & -4 \\ 12 & -4 & -3 \end{pmatrix}$$

$$= \det \begin{pmatrix} 8 & -4 \\ -4 & -3 \end{pmatrix}_{2\times 2} \vec{i} - \det \begin{pmatrix} 1 & -4 \\ 12 & -3 \end{pmatrix} \vec{j} + \det \begin{pmatrix} 1 & 8 \\ 12 & -4 \end{pmatrix} \vec{k}$$

$$= -40\vec{i} - 45\vec{j} - 100\vec{k} = \begin{pmatrix} -40 \\ -45 \\ -100 \end{pmatrix}.$$

2^{nd} step: $\vec{a} \times (\vec{b} \times \vec{c}) = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \times \begin{pmatrix} -40 \\ -45 \\ -100 \end{pmatrix}$

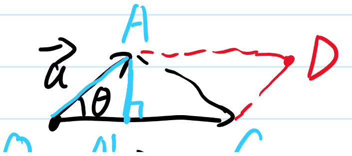
$$= \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -2 & 1 \\ -40 & -45 & -100 \end{pmatrix}$$

$$= \det \begin{pmatrix} -2 & 1 \\ -45 & -100 \end{pmatrix} \vec{i} - \det \begin{pmatrix} 2 & 1 \\ -40 & -100 \end{pmatrix} \vec{j}$$

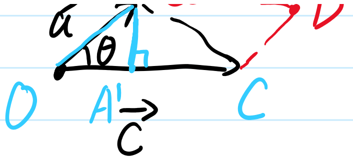
$$+ \det \begin{pmatrix} 2 & -2 \\ -40 & -45 \end{pmatrix} \vec{k}$$

$$= 245\vec{i} + 160\vec{j} - 170\vec{k}.$$

(e) area of triangle with $\vec{a}$ and $\vec{c}$.



area of $\triangle = \frac{1}{2}\left|\vec{a} \times \vec{c}\right|$



$$\text{area of } \triangle = \frac{1}{2}\underbrace{|\vec{a}\times\vec{c}|}$$

$$= \frac{1}{2}\underbrace{|\vec{a}|}\cdot|\vec{c}|\cdot\underline{\sin\theta}$$

$$= \frac{1}{2}\underbrace{|\overrightarrow{AA'}|\cdot|\vec{c}|}$$

$$\boxed{}OADC = |\vec{a}\times\vec{c}|$$

(f) $\underbrace{\vec{a}\cdot\vec{b}}_{2^{n}}\underset{\text{1st}}{\times\vec{c}} = \begin{pmatrix}2\\-2\\1\end{pmatrix}\cdot\begin{pmatrix}-40\\-45\\-100\end{pmatrix}$

if scalar × vector

make no sense !!!

$$= 2\times(-40)+(-2)\times(-45)+1\times(-100)$$

$$= -80+90-100 = -90.$$

(g) volume of tetrahedron with $\vec{a}+\vec{c}$, $\vec{a}-\vec{c}$, $\vec{b}$.

$$= \frac{1}{6}\underbrace{(\vec{a}+\vec{c})}\cdot\underbrace{(\vec{a}-\vec{c})\times\vec{b}}.$$

$$\vec{a}+\vec{c} = \begin{pmatrix}2\\-2\\1\end{pmatrix}+\begin{pmatrix}12\\-4\\-3\end{pmatrix}=\begin{pmatrix}14\\-6\\-2\end{pmatrix}= 14\vec{i}-6\vec{j}-2\vec{k}.$$

$$\vec{a}-\vec{c} = \begin{pmatrix}2\\-2\\1\end{pmatrix}-\begin{pmatrix}12\\-4\\-3\end{pmatrix}=\begin{pmatrix}-10\\2\\4\end{pmatrix}= -10\vec{i}+2\vec{j}+4\vec{k}.$$

$$(\vec{a}-\vec{c})\times\vec{b} = \begin{pmatrix}-10\\2\\4\end{pmatrix}\times\begin{pmatrix}1\\8\\-4\end{pmatrix}$$

$$= \det\begin{pmatrix}\vec{i} & \vec{j} & \vec{k}\\ -10 & 2 & 4\\ 1 & 8 & -4\end{pmatrix}.$$

$$= \det\begin{pmatrix}2 & 4\\ 8 & -4\end{pmatrix}\vec{i} - \det\begin{pmatrix}-10 & 4\\ 1 & -4\end{pmatrix}\vec{j} + \det\begin{pmatrix}-10 & 2\\ 1 & 8\end{pmatrix}\vec{k}$$

$$= -40\vec{i} - 36\vec{j} - 82\vec{k}$$

$= -40\vec{i} - 36\vec{j} - 82\vec{k}.$

$$V = \frac{1}{6} \begin{pmatrix} 14 \\ -6 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} -40 \\ -36 \\ -82 \end{pmatrix} = \frac{1}{6} \times \left| 14 \times (-40) + (-6) \times (-36) + (-2) \times (-82) \right|.$$

$$= \frac{1}{6} \times 180$$

$$= 30 \quad \#$$

P3 (Assignment 1)

$P(1,1,1)\ Q(2,1,0)$

$$\overrightarrow{PQ} = (2,1,0) - (1,1,1)$$

$$= (1,0,-1)$$

$$= \vec{i} - \vec{k}.$$

$$|\overrightarrow{PQ}| = \sqrt{1^2 + 0^2 + (-1)^2} = \sqrt{2}.$$

unit vector $\widehat{PQ}$

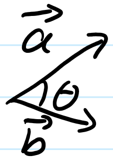
$$\widehat{PQ} = \frac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|} = \frac{1}{\sqrt{2}}(\vec{i} - \vec{k})$$

$$= \frac{1}{\sqrt{2}}\vec{i} - \frac{1}{\sqrt{2}}\vec{k}.$$

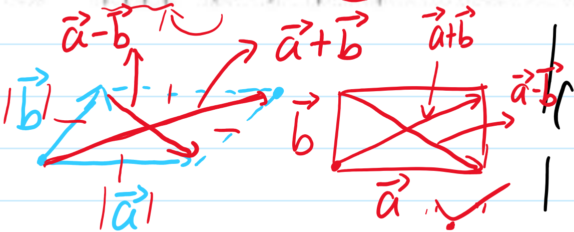
2. Show that for any vectors $\vec{a}$ and $\vec{b}$

(a) $\left|\vec{a}+\vec{b}\right|^2 + \left|\vec{a}-\vec{b}\right|^2 = 2\left(\left|\vec{a}\right|^2 + \left|\vec{b}\right|^2\right)$; ✓

(b) $\left|\vec{a}\right|^2\left|\vec{b}\right|^2 = \left(\vec{a}\cdot\vec{b}\right)^2 + \left|\vec{a}\times\vec{b}\right|^2$; [Hint $\cos^2\theta + \sin^2\theta \equiv 1$] ✓



(a)



$$|\vec{a}+\vec{b}|^2 = (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})$$

(b) $$(\vec{a}\cdot\vec{b})^2 = (|\vec{a}|\cdot|\vec{b}|\cdot\cos\theta)^2 \quad ③$$

$$= |\vec{a}|^2\cdot|\vec{b}|^2\cdot\cos^2\theta.$$

$$|\vec{a}\times\vec{b}|^2 = (|\vec{a}|\cdot|\vec{b}|\cdot\sin\theta)^2 \quad (4)$$

$$|\vec a+\vec b|^2 = (\vec a+\vec b)\cdot(\vec a+\vec b)$$

① $$= \vec a\cdot\vec a+\vec a\cdot\vec b+\vec b\cdot\vec a+\vec b\cdot\vec b$$
$$= |\vec a|^2 + 2\vec a\cdot\vec b + |\vec b|^2$$

$$|\vec a-\vec b|^2 = (\vec a-\vec b)\cdot(\vec a-\vec b)$$

② $$= \vec a\cdot\vec a-\vec a\cdot\vec b-\vec b\cdot\vec a+\vec b\cdot\vec b$$
$$= |\vec a|^2 - 2\vec a\cdot\vec b + |\vec b|^2$$

$$①+② = 2(|\vec a|^2+|\vec b|^2)$$

$$|a\times b| = (|\vec a|\cdot|\vec b|\cdot\sin\theta)$$
vector
$$= |\vec a|^2\cdot|\vec b|^2\cdot\sin^2\theta.$$ ④

$$③+④ = |\vec a|^2\cdot|\vec b|^2(\underbrace{\cos^2\theta+\sin^2\theta}_{=1})$$
$$= |\vec a|^2\cdot|\vec b|^2.$$

3. Given points $P(1,1,1)$ and $Q(2,1,0)$ find the vector $\overline{PQ}$, the length of $\overline{PQ}$ and a unit vector in the same direction as $\overline{PQ}$.

$$\overrightarrow{PQ} = \overrightarrow{OQ}-\overrightarrow{OP} = (2,1,0)-(1,1,1) = (1,0,-1)$$

$O=(0,0,0)$ origin point

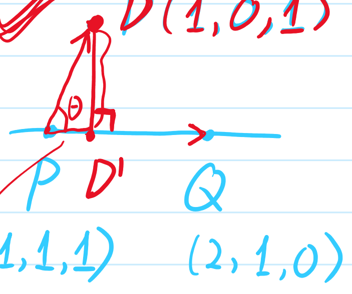
$$= \vec i-\vec k$$

Length: $|\overrightarrow{PQ}| = \sqrt{\overrightarrow{PQ}\cdot\overrightarrow{PQ}} = \sqrt{1^2+0^2+(-1)^2} = \sqrt{2}.$ ✓

Unit vector: $\widehat{PQ} = \dfrac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|} = \dfrac{\vec i-\vec k}{\sqrt2} = \dfrac{1}{\sqrt2}\vec i-\dfrac{1}{\sqrt2}\vec k. = \begin{pmatrix}\frac{1}{\sqrt2}\\0\\-\frac{1}{\sqrt2}\end{pmatrix}.$

$|DD'|$

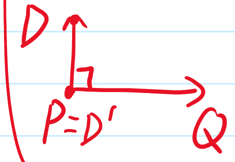
4. Find the shortest distance between the point $D(1,0,1)$ and the line that passes through the points $P(1,1,1)$ and $Q(2,1,0)$.



Solve a triangle

$|\overrightarrow{PD}|\cdot\sin\theta$

$$\overrightarrow{PD} = \overrightarrow{OD}-\overrightarrow{OP} = (1,0,1)-(1,1,1) = (0,-1,0)$$
$$\overrightarrow{PQ} = \overrightarrow{OQ}-\overrightarrow{OP} = (2,1,0)-(1,1,1) = (1,0,-1)$$
$$\overrightarrow{PD}\cdot\overrightarrow{PQ} = 0\times1+(-1)\times0+0\times(-1) = 0$$



$$|\overrightarrow{PD}| = \sqrt{0^2+(-1)^2+0^2} = 1.$$

$$DD' = \sqrt{|\overrightarrow{PD}|^2-|PD'|^2}$$

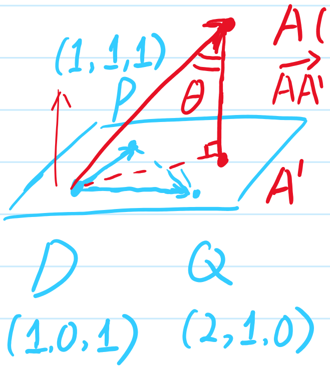
✓

$$|PD'| = |\overrightarrow{PD}|\cdot\cos\theta = \cancel{|\overrightarrow{PD}|}\cdot\frac{\overrightarrow{PD}\cdot\overrightarrow{PQ}}{\cancel{|\overrightarrow{PD}|}\cdot|\overrightarrow{PQ}|} = \frac{\overrightarrow{PD}\cdot\overrightarrow{PQ}}{|\overrightarrow{PQ}|}$$

$$|\vec{PD'}| = |\vec{PD}| \cdot \cos\theta = \cancel{|\vec{PD}|} \cdot \frac{\vec{PD}\cdot\vec{PQ}}{\cancel{|\vec{PD}|}\cdot|\vec{PQ}|} = \frac{\vec{PD}\cdot\vec{PQ}}{|\vec{PQ}|}.$$ ✓

$|AA'|$

6. Hence find the shortest distance between the point $A(-1,0,2)$ and the plane containing $D(1,0,1)$, $P(1,1,1)$ and $Q(2,1,0)$.



$A(-1,0,2)$

$\vec{AA'} \uparrow \vec{n}$

$\vec{DP} = \vec{OP} - \vec{OD} = (1,1,1) - (1,0,1) = (0,1,0)$

$\vec{DQ} = \vec{OQ} - \vec{OD} = (2,1,0) - (1,0,1) = (1,1,-1)$

Solve a triangle $\triangle ADA'$

$DA = (-1,0,2) - (1,0,1) = (-2,0,1)$ ✓

$$|\vec{DA}| \cdot \cos\theta = \cancel{|\vec{DA}|} \cdot \frac{\vec{DA}\cdot\vec{A'A}}{\cancel{|\vec{DA}|}\cdot|\vec{A'A}|} = \frac{\vec{DA}\cdot\vec{A'A}}{|\vec{A'A}|} = \vec{DA}\cdot\vec{n}$$

$\vec{n}$: unit vector in direction of $A'A$.

$\parallel$

✓ $\vec{DP} \times \vec{DQ}$ $\quad \det\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} = a_1a_4 - a_2a_3$

$$= \det\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}$$

$$= \det\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}\vec{i} - \det\begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}\vec{j} + \det\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\vec{k}$$

$0\times1 - 1\times1$

$$= -\vec{i} - \vec{k} = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$$

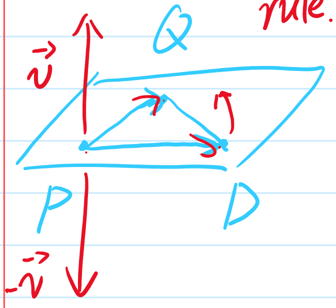
$$\vec{n} = \frac{-\vec{i} - \vec{k}}{\sqrt{(-1)^2 + 0^2 + (-1)^2}} = -\frac{1}{\sqrt{2}}\vec{i} - \frac{1}{\sqrt{2}}\vec{k} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

$$\Rightarrow \vec{DA}\cdot\vec{n} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}\cdot\begin{pmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = \frac{\sqrt{2}}{2}$$ #.

5. Find a unit vector that is perpendicular to plane that contains the points $D(1,0,1)$, $P(1,1,1)$ and $Q(2,1,0)$.

all

right-hand rule.



$\vec{PD} = \vec{OD} - \vec{OP} = (1,0,1) - (1,1,1) = (0,-1,0)$

$\vec{PQ} = \vec{OQ} - \vec{OP} = (2,1,0) - (1,1,1) = (1,0,-1)$

$\vec{v} = \vec{PD} \times \vec{PQ}$ $\quad -(\vec{PD} \times \vec{PQ}) = \vec{PQ} \times \vec{PD}$

$$= \det\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

$$= \det\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\vec{i} - \det\begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}\vec{j} + \det\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\vec{k}.$$

$= \vec{i} + \vec{k}$ A B C D $\longrightarrow$ $\vec{n} = \frac{\vec{v}}{|\vec{v}|} = \frac{\vec{i} + \vec{k}}{\sqrt{1^2 + 0^2 + 1^2}}$

7. Find the value of k such that (3, −3, 2), (1, 0, 1), (1, 1, 0) and (0, 1, k) are coplanar.

$= \vec{i}+\vec{k}$ A B C D $\longrightarrow \vec{n} = \frac{\vec{v}}{|\vec{v}|} = \frac{\vec{i}+\vec{k}}{\sqrt{1^2+0^2+1^2}}$

$= \frac{1}{\sqrt{2}}\vec{i}+\frac{1}{\sqrt{2}}\vec{k}$ #.

7. Find the value of k such that (3, -3, 2), (1, 0, 1), (1, 1, 0) and (0, 1, k) are coplanar.

unknown variable.

$\vec{AB} = \vec{OB} - \vec{OA} = (1,0,1) - (3,-3,2$

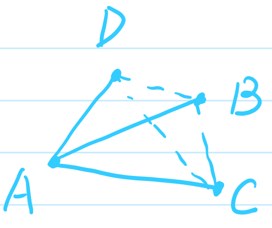
$= (-2, 3, -1)$

$\vec{AC} = \vec{OC} - \vec{OA} = (1,1,0) - (3,-3,2)$

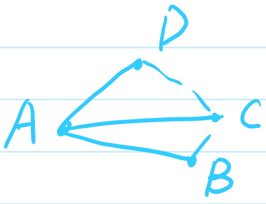
$= (-2, 4, -2)$

$\vec{AD} = \vec{OD} - \vec{OA} = (0,1,k) - (3,-3,2)$

$= (-3, 4, k-2)$.



(X)



(✓)

$\vec{AD} \cdot (\vec{AB} \times \vec{AC}) = 0$ coplanar ✓

$\neq 0$ tetrahedron.

step I: $\vec{AB} \times \vec{AC}$

$= \det\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 3 & -1 \\ -2 & 4 & -2 \end{pmatrix}$

$= \det\begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix}\vec{i} - \det\begin{pmatrix} -2 & -1 \\ -2 & -2 \end{pmatrix}\vec{j} + \det\begin{pmatrix} -2 & 3 \\ -2 & 4 \end{pmatrix}\vec{k}$.

$= -2\vec{i} - 2\vec{j} - 2\vec{k} = \begin{pmatrix} -2 \\ -2 \\ -2 \end{pmatrix}$.

step II: $\vec{AD} \cdot (\vec{AB} \times \vec{AC}) = 0$

$\Rightarrow \begin{pmatrix} -3 \\ 4 \\ k-2 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ -2 \\ -2 \end{pmatrix} = 0$

$\Rightarrow (-3)\times(-2) + 4\times(-2) + (k-2)\times(-2) = 0$

$\Rightarrow k = 1$ #

MA0101 Tutorial Class TB3 session.

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Chapter 2 Differentiation:

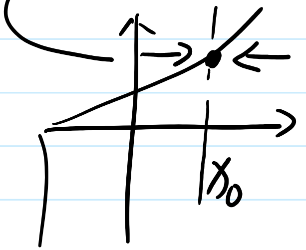
① Basic Definition: "limit": $f'(x_0)=\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$

$h>0$, $h\to 0^+$

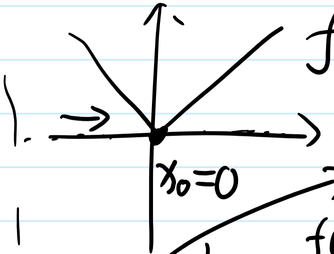
$h<0$, $h\to 0^-$

② known results: ✓

$f(x)$	$f'(x)=\frac{df}{dx}$
x^a	ax^{a-1}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
e^x	e^x
$\ln x$	$\frac{1}{x}$
constant c	0



left limit = right limit



$f(x)=|x|$

$f'(0)\neq$ NOT exist.

$\lim_{h\to0^+}\frac{f(0+h)-f(0)}{h}\to 1$.

$\lim_{h\to0^-}\frac{f(0+h)-f(0)}{h}\to -1$.

Rules: where complicated?

✓ Product: $f(x)g(x)$

✓Product: $f(x)g(x)$

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$$

$$\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} \to -1.$$

$f(x)=|x|$, $f'(x)$ NOT exist

✓Quotient rule: $\frac{f(x)}{g(x)}$.

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

✓Addition rule: $af(x)+bg(x)$

$$\frac{d}{dx}[af(x)+bg(x)] = af'(x) + bg'(x)$$

✓Chain rule: $f(g(x))$

$$\frac{d}{dx}[f(g(x))] = f'(g(x))\cdot g'(x).$$

1.(a) $\frac{d}{dx}[4e^x+2x+\sin x]$

$= 4\frac{d}{dx}(e^x) + 2\frac{d}{dx}(x) + \frac{d}{dx}(\sin x)$

$= 4e^x + 2 \quad + \cos x$

$(x^a)' = a\cdot x^{a-1}$ ✓

$(x^1)' = 1\cdot x^{1-1} = 1$

(b) $\frac{d}{dx}(3\cos x + 4x^7 - 5\ln x).$

$= 3\frac{d}{dx}(\cos x) + 4\frac{d}{dx}(x^7) - 5\frac{d}{dx}(\ln x).$

$= 3(-\sin x) + 4\times 7\times x^6 - 5\frac{1}{x}$

$= -3\sin x + 28x^6 - \frac{5}{x}.$

$= -3\sin x + 28x^6 - \frac{5}{x}.$

(c) $\frac{d}{dx}(\sin x \cdot \ln x)$

$= \frac{d}{dx}(\sin x)\cdot \ln x + \sin x \cdot \frac{d}{dx}(\ln x)$ Step I. take rule!

$= \cos x \cdot \ln x + \sin x \cdot \frac{1}{x}.$

(d) $\frac{d}{dx}(\underline{x^2 e^x}).$

$= \frac{d}{dx}(x^2)e^x + x^2 \underline{\frac{d}{dx}(e^x)}$

$= 2x \cdot e^x + x^2 \cdot \underline{\underline{e^x}}$

(e) $\frac{d}{dx}(\underbrace{e^{\boxed{x^2}}}_{f(x)}) \Rightarrow$ let $g(x) = x^2$, $f(x) = e^{g(x)}$

$f[g(x)] = e^{g(x)}$

$\boxed{\text{chain rule}}$

$= f'(g(x)) \cdot g'(x)$

$= e^{g(x)} \cdot \frac{d}{dx}(x^2)$

$= e^{x^2} \cdot 2x.$

(f) $\frac{d}{dx}(\underbrace{\cos(\boxed{2e^x+1})}_{f(x)}) \Rightarrow$ let $g(x) = \underline{2e^x+1}.$

$f(g(x)) = \cos g(x)$

$\boxed{\text{chain rule.}}$

$= f'(g(x)) \cdot g'(x)$

$= -\sin(g(x)) \cdot \frac{d}{dx}(2e^x+1)$

$= -\sin(2e^x+1)\cdot(2e^x+0)$

$$= -\sin(2e^x+1)\cdot(2e^x \cancel{+0})$$

$$= -2e^x \sin(2e^x+1)$$

(g) $\frac{d}{dx}\left(\frac{x-1}{x+1}\right)$ $\frac{f(x)}{g(x)}$

$$= \frac{(x-1)'\cdot(x+1) - (x-1)\cdot(x+1)'}{(x+1)^2}$$

$$= \frac{1\cdot(\cancel{x}+1) - (\cancel{x}-1)\cdot 1}{(x+1)^2}$$

$$= \frac{2}{(x+1)^2}$$

stepI:

take which rule!

stepII

(h) $\frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$ $\frac{f(x)}{g(x)}$

$$= \frac{(\sin x)'\cdot\cos x - \sin x\cdot(\cos x)'}{\cos^2 x}$$

$$= \frac{\cos x\cdot\cos x - \sin x\cdot(-\sin x)}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$ $\ln x^{x^2}$ (x^x)

(i) $f(x) = x^{x^2}$ ✓ $\Longrightarrow$ let $g(x)=x^2$, $f[g(x)] = \boxed{x^{g(x)}}$

$f'(x)$

$\Downarrow$ stepI: take ln.

$\ln f(x) = x^2\cdot\ln x$

$(x^{a})'$ ✓

$(e^{g(x)})'$ ✓

$\Downarrow$ stepII: differentation.

Chain

⇓ step II: differentation. $(e^{g(x)})'$ ✓

chain ⇓ ⇓ product.

$$\frac{1}{f(x)} \cdot \boxed{f'(x)} = (x^2)' \cdot \ln x + x^2 (\ln x)'$$

$$f'(x) = \left(2x \cdot \ln x + \cancel{x^2} \cdot \frac{1}{\cancel{x}}\right) x^{x^2}$$

$$f'(x) = (2x\ln x + x)\, x^{x^2}.$$ ✓

(j). $f(x) = x^{2\ln x}$ $\longrightarrow \ln x^{2\ln x}$

⇓ ln.

$$\ln f(x) = 2\ln x \cdot \ln x = 2(\ln x)^2$$

⇓ differentiation

$$\frac{1}{f(x)} \cdot f'(x) = 2 \cdot 2\ln x \cdot (\ln x)'$$

$$f'(x) = 4\ln x \cdot \frac{1}{x} \cdot x^{2\ln x}$$

$$f'(x) = 4\ln x \cdot x^{2\ln x - 1}.$$

(k) $f(x) = x \cdot \ln(x^2+1)$

$$f'(x) = (x)' \cdot \ln(x^2+1) + x \cdot [\ln(x^2+1)]'$$

step I product

$g(x) = x^2+1$

$(\ln g(x))'$

$$= 1 \cdot \ln(x^2+1) + x \cdot \frac{1}{g(x)} \cdot g'(x)$$

$$= \ln(x^2+1) + x \cdot \frac{1}{x^2+1} \cdot 2x$$

step II

$g'(x) = (x^2+1)' = 2x$

chain rule.

$$= \ln(x^2+1) + \frac{2x^2}{x^2+1}$$ #.

② (a) Find |gradient| of implicit curve $x^3+y^3=a$

② a) Find $\boxed{\text{gradient}}$ of implicit curve $x^3 + y^3_{(x)} = 9$. at $(x,y) = (2,1)$

$$\boxed{\frac{dy}{dx}}_{(x,y)} \Rightarrow 3x^2 + 3y^2 \cdot \boxed{\frac{dy}{dx}} = 0$$

$$\Rightarrow \boxed{\frac{dy}{dx}} = -\frac{x^2}{y^2}$$

Substitute $(2,1)$ $\nearrow$ $\frac{dy}{dx}(2,1) = -\frac{2^2}{1^2} = -4$.

b) Find points (x,y) at which $\boxed{x^2 - x \cdot y_{(x)} + y^2_{(x)} = 3}$ has zero slope.

"$\frac{dy}{dx} = 0$" $\Rightarrow 2x - [(x)' \cdot y(x) + x\frac{dy}{dx}] + 2y \cdot \frac{dy}{dx} = 0$

Method I: $\Rightarrow 2x - y(x) - x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$

$$\Rightarrow \frac{dy}{dx} = \frac{y - 2x}{2y - x} = 0$$

$$\Rightarrow y = 2x$$

$$\begin{cases} y = 2x \\ x^2 - xy + y^2 = 3 \end{cases} \Rightarrow \begin{matrix} x^2 - 2x^2 + 4x^2 = 3 \\ x = \pm 1. \end{matrix} \nearrow \begin{matrix} y = \pm 2 \\ (1,2) \text{ and } (-1,-2) \end{matrix}$$

Method II (Extensions): insight of Geometric

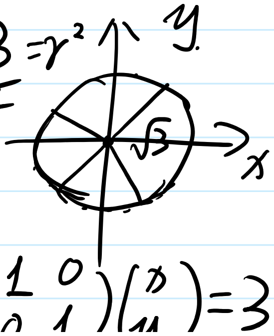
$$(x\ y)\underbrace{\begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix}}\begin{pmatrix} x \\ y \end{pmatrix} = 3$$

$$\Downarrow$$

$$(x\ y)\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 3.$$

$x^2 + y^2 = 3 = r^2$

$$\Downarrow$$



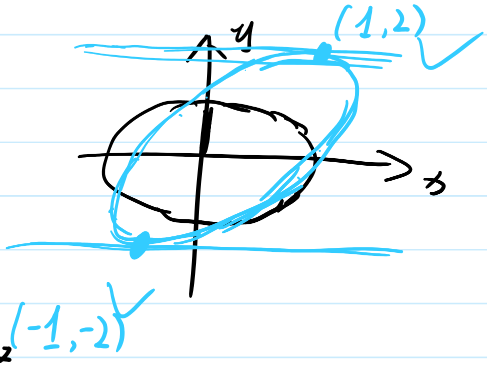
$$(x\ y)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 3$$

$$(x \ y)\begin{pmatrix} & \\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}\begin{pmatrix} & \\ 0 & 1\end{pmatrix}\begin{pmatrix} & \\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}\begin{pmatrix} \\ y\end{pmatrix} = 3.$$

$\Downarrow$ rotation $\Downarrow$ enlarge. $\Downarrow$ rotation

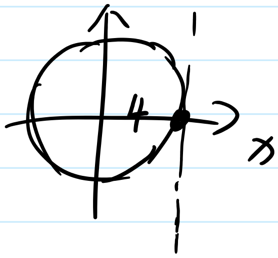
$$\begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}$$

$$(x \ y)\underbrace{\begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}}\begin{pmatrix}x \\ y\end{pmatrix} = 3$$



C). gradient of curve $\underline{x^2+y^2=16=r^2}$

(I) $(x,y)=(4,0)$



"∞" (II)

$$2x+2y\cdot\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=-\frac{x}{y}$$

$$(4,0) \quad =-\frac{4}{0}$$

Nonsense

$\frac{dy}{dx}(4,0)$ Does NOT exist.

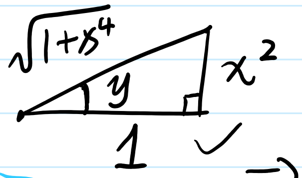
---

3) $y=\underbrace{\tan}_{\theta}{}^{-1}\underbrace{(x^2)}_{\Delta}$. $\frac{dy}{dx}$

Method I: $\underbrace{\tan y}_{\theta}=\underbrace{x^2}_{\Delta}$

$$\Rightarrow \frac{1}{\cos^2 y}\cdot\frac{dy}{dx}=2x$$

$$\Rightarrow \frac{dy}{dx}=2x\cdot\cos^2 y$$



$$\cos y=\frac{1}{\sqrt{1+x^4}}$$

$$\Rightarrow \cos^{2} y=\frac{1}{1+x^4}$$

$$(\tan z)'=\frac{1}{\cos^2 z}$$

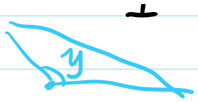
known result.

Method II: Chain rule

$$\frac{dy}{dx}=\frac{1}{1+x^4}\cdot 2x=\frac{2x}{1+x^4}$$

$$(\tan^{-1} z)'=\frac{1}{1+z^2}$$

known result.



$$\Rightarrow \cos^{2}y = \frac{1}{1+x^4}$$

$$\Rightarrow \frac{dy}{dx} = 2x\frac{1}{1+x^4}$$

b) $y = \textcircled{2}\cos^{-1}(\ln x)$ ✓ $\quad \frac{dy}{dx} \quad y \in [\ \ ]$

Method I: $\cos\frac{y(x)}{2} = \ln x$

$\cos^{-1}x + \sin^{-1}x = \frac{\pi}{2}$

$$\Rightarrow -\sin\frac{y}{2}\cdot\frac{1}{2}\cdot\frac{dy}{dx} = \frac{1}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2}{x\sin\frac{y}{2}}.$$

$\textcircled{\sin\frac{y}{2}}$

$$\cos^2\frac{y}{2} + \sin^2\frac{y}{2} = 1$$

$$\Rightarrow \sin\frac{y}{2} = \pm\sqrt{1-\cos^2\frac{y}{2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2}{x(\pm\sqrt{1-\cos^2\frac{y}{2}})}.$$ ✓

result.

Method II: $\boxed{\text{Chain rule.}}$

$$\frac{dy}{dx} = 2\left(-\frac{1}{\sqrt{1-(\ln x)^2}}\right)\cdot\frac{1}{x}$$

$$\left((\cos^{-1}z)' = -\frac{1}{\sqrt{1-z^2}}\right)$$

~~known result~~.

$$= -\frac{2}{x\sqrt{1-(\ln x)^2}}.$$

$\boxed{4}$ $\boxed{\text{linear approximation}}$ 1st-order $\quad f(x) = \ln x \quad$ at $\boxed{x=1}$.

Taylor series $\quad f(x) = f(1) + f'(1)(x-1) + \frac{1}{2!}f''(1)(x-1)^2 + \cdots$

$\underbrace{f(1)}_{0}$

$f^{(1)}(x) = (\ln x)' = \frac{1}{x} = x^{-1}$ odd $\textcircled{x^9}$ $\Rightarrow f'(1) = 1$

$f^{(2)}(x) = (-1)x^{-2}$ even $\Rightarrow f^{(2)}(1) = -1$

$f^{(3)}(x) = (-1)(-2)x^{-3}$ odd $\Rightarrow f^{(3)}(1) = 2!$

$\vdots$

$$f^{(n)}(x) = (-1)^{n-1}(n-1)!\,x^{-n} \Rightarrow f^{(n)}(1) = (-1)^{n-1}(n-1)!$$

$$f(x) \approx 0 + (x-1) \cancel{+} \frac{1}{2!}(-1)(x-1)^2 + \cdots$$

$$\approx \sum_{n=1}^{\infty} (-1)^{n-1}(n-1)!\,(x-1)^n$$

5、 quadratic approximation $\boxed{f(x) = e^{2x}}$ at $\boxed{x=0}$.

$$f(x) = f(0) + f'(0)x + \frac{1}{2!}f''(0)x^2 \cancel{+} \cdots$$

$$f'(x) = 2e^{2x} \Rightarrow f'(0) = 2$$

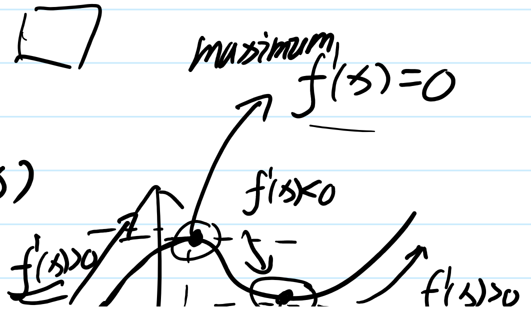
$$f''(x) = 2\cdot 2e^{2x} \Rightarrow f''(0) = 4.$$

$$f(x) \approx 1 + 2x + 2x^2$$

	true value $f(x) = e^{2x}$	approximated value $f(x) \approx 1+2x+2x^2$	
$x=0.1$	1.22	1.22	✓
$x=0.25$	1.65	1.63	✓
$x=2$	54.60	13.00.	☐

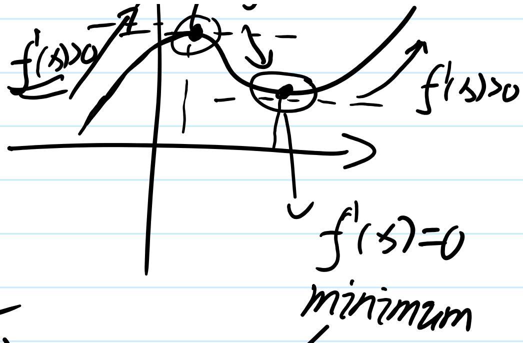
6. (a) $2x^5 + 11x^4 - 10x^3 + 17 = f(x)$

$f'(x) = 0$



$=$

$\checkmark \underline{f'(x)=0}$

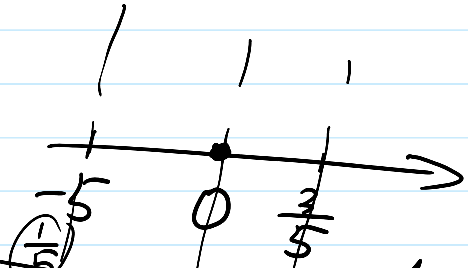


step I: $\underline{f'(x)} = \underline{10}x^{④} + \underline{44}x^3 - \underline{30}x^2 = 0$

$\Longrightarrow 2x^2(\underline{5x^2+22x-15}) = 0$

$\Longrightarrow \underline{2x^2(x+5)(5x-3)} = 0.$

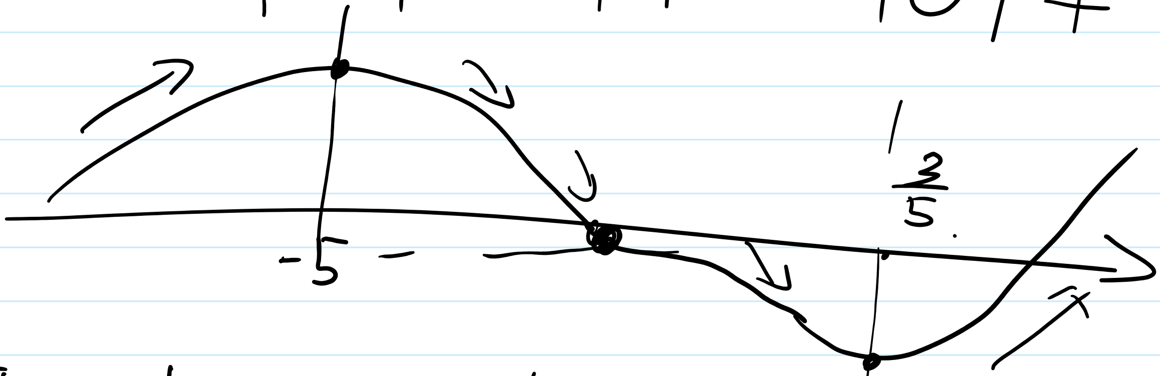
$x_1 = x_2 = 0$, $x_3 = -5$, $x_4 = \frac{3}{5}.$



step II: $\boxed{x=-6}$

x	$(-\infty,-5)$	-5	$(-5,0)$	0	$(0,\frac{3}{5})$	$\frac{3}{5}$	$\frac{3}{5},+\infty$
$f(x)$	↗		↘		↘		↗
$f'(x)$	$+$	0	$-$	0	$-$	0	$+$

1



step III: conclusion: $f(x)$ has maximum at $x=-5.$
has minimum at $x=\frac{3}{5}.$

(b) $f(x) = x^{\frac{2}{3}}\cdot(2x-1)$

$f'(x) = (\underbrace{x^{\frac{2}{3}}}_{\frac{2}{3}x^{-\frac{1}{3}}})'(2x-1) + x^{\frac{2}{3}}\underbrace{(2x-1)'}_{2} = \underline{\frac{10}{3}x^{\frac{2}{3}} - \frac{2}{3}x^{-\frac{1}{3}}} = 0$

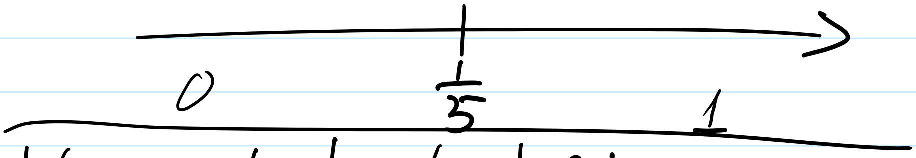
step I:

stepI: $\frac{2}{3}x^{-3}$ 2

$\Rightarrow \underline{5x^{\frac{2}{3}} - x^{-\frac{1}{3}} = 0}$ $(\times x^{\frac{1}{3}})$

$\Rightarrow 5x - 1 = 0$

$\Rightarrow x = \frac{1}{5}.$



StepII:

x	$(-\infty, \frac{1}{5})$	$\frac{1}{5}$	$(\frac{1}{5}, +\infty)$
$f(x)$	↘		↗
$f'(x)$	$-$	0	$+$

stepIII: <u>Conclusion</u>: $f(x)$ has minimum at $x = \frac{1}{5}.$

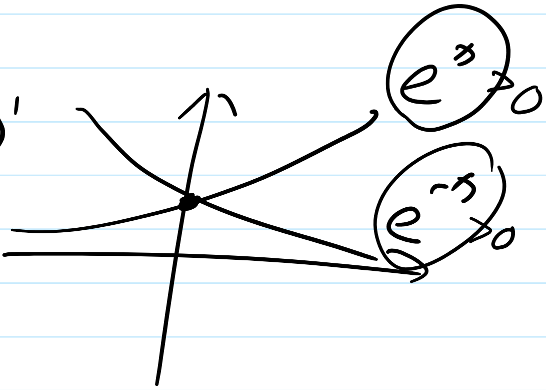
(C) $x \cdot e^{-\frac{x^2}{2}} = f(x)$

$f'(x) = (x)' e^{-\frac{x^2}{2}} + x \cdot (e^{-\frac{x^2}{2}})'$

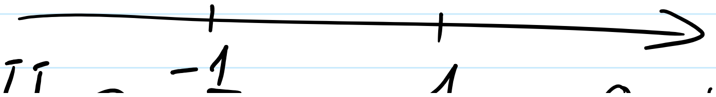
stepI: $= e^{-\frac{x^2}{2}} + x \cdot e^{-\frac{x^2}{2}} \cdot (-\frac{x^2}{2})'$

$= e^{-\frac{x^2}{2}} + x e^{-\frac{x^2}{2}} \cdot (-x)$

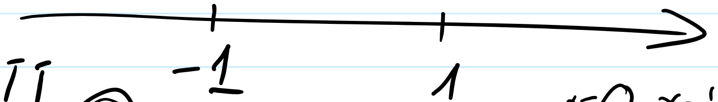
$= \underline{(1-x^2)\underbrace{e^{-\frac{x^2}{2}}}_{>0} = 0}$



$\Rightarrow 1 - x^2 = 0 \Rightarrow x_1 = 1,\ x_2 = -1.$ ✓



stepII



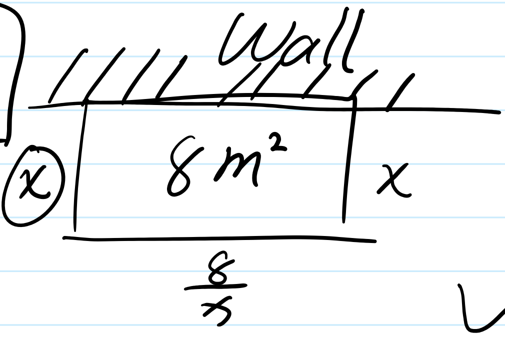
step II ⓶ $x=0, x=\frac{1}{2}$ $x=2$

x	$(-\infty, -1)$	-1	$(-1, 1)$	1	$(1, +\infty)$
$f(x)$	↘	$-e^{-\frac{1}{2}}$	↗	$e^{-\frac{1}{2}}$	↘
$f'(x)$	$-$	0	$+$	0	$-$
	$f'(-2)<0$	min	$f'(0)=1>0$	max	$f'(2)<0$

step III: conclusion: $f(x)$ has minimum at $x=-1$

maximum at $x=1$.

7.



minimum length of fencing required.

✓ total length: $2x+\frac{8}{x}=f(x)$

$$f'(x)=2-\frac{8}{x^2}=0$$

$$\Rightarrow 2x^2=8 \Rightarrow x^2=4 \Rightarrow \boxed{x_1=2} \text{ or } \cancel{x_2=-2}.$$

x	0	$(0,2)$	2	$(2,+\infty)$
$f(x)$		↘	8	↗
$f'(x)$	\	$-$	0	$+$

local minimum

Conclusion: $f(x)$ attains its minimum at $x=2$.

where the length of fence is

Conclusion: $f(x)$ attains its minimum at $x=2$.
where the length of fence is 8 m.

Chapter 3

Monday, March 16, 2020 4:44 PM

MA0101 Tutorial Class (TB) session.

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Integration ⟺ differentiation

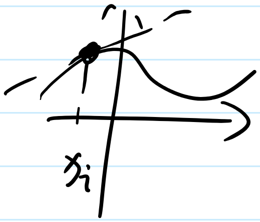
straightforward

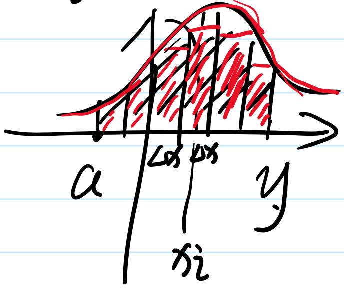
fundamental theorem of Calculus.

$F(y) = \int_a^y f(x)dx$

$= \lim_{\Delta x_i \to 0} \sum f(x_i)\Delta x_i$

$f'(x_i)$





✓

Integration table. ✓

$f(x)$ ⟷	$\int f(x)dx$
$\frac{1}{x}$ ✓	$\ln x + C$

$\frac{1}{x}$ ✓	$\ln x + C$
x^n	$\frac{1}{n+1}x^{n+1} + C$
e^x	$e^x + C$
$\sin x$	$-\cos x + C$
$\cos x$	$\sin x + C$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1}(x) + C$
$\frac{1}{1+x^2}$	$\tan^{-1}(x) + C.$

$$\longrightarrow \int x\,dx = \frac{1}{2}x^2 + C$$

$$\int x^7 dx = \frac{1}{8}x^8 + C.$$

$$\int x^{-\frac{1}{2}} dx = \frac{1}{-\frac{1}{2}+1} x^{-\frac{1}{2}+1} + c$$

$$= 2x^{\frac{1}{2}} + c$$

1. (a) $\int 4e^x + 2x + \sin x\, dx.$

$$= 4\int \underline{e^x} dx + 2\underline{\int x\,dx} + \int \sin x\, dx$$

$$= 4(e^x + \underline{C_1}) + 2(\tfrac{1}{2}x^2 + \underline{C_2}) + (-\cos x + \underline{C_3})$$

$$= 4e^x + x^2 - \cos x + (\underbrace{4C_1 + 2C_2 + C_3}_{\substack{\shortparallel \\ C}})$$

(b) $\int \underline{3\cos x} + \underline{4x^7} dx$

$$= 3\int \underline{\cos x\, dx} + 4\int x^7 dx.$$

$$= 3(\sin x + C_1) + 4(\tfrac{1}{8}x^8 + C_2)$$

$$= 3\sin x + \frac{1}{2}x^8 + (\underbrace{3C_1 + 4C_2}_{\substack{\shortparallel \\ C}})$$

(c) $\int (\sin x - 5\frac{1}{x}) dx$

$\ddot{C}$

(c) $\int(\sin x - 5\frac{1}{x})dx$

$= \int \underline{\sin x}\, dx - 5\int \frac{1}{x} dx$

$= -\cos x + C_1 - 5(\ln x + C_2)$

$= -\cos x - 5\ln x + \underbrace{(C_1 - 5C_2)}_{\overset{\shortparallel}{C}}.$

2. Substitution.

(a) $\int\left(4e^{\boxed{2x}} + 6\sin(\boxed{2x+1})\right)dx$

$\int e^{\boxed{x}}\, d\boxed{x} = e^{x} + C$

$= 4\underline{\int e^{\boxed{2x}}\, \underline{dx}} + 6\int \sin(2x+1)\, \underline{dx}.$

$\cancel{\int e^{2x} dx \overset{?}{=} e^{2x} + C}$

$\cancel{e^{2x}+C}$

let $y = 2x$

$\Rightarrow dy = 2\,dx \Rightarrow dx = \frac{1}{2}dy$

$\Downarrow$

let $z = \boxed{2x+1}$

$\Rightarrow dz = 2dx$

$\Rightarrow dx = \frac{1}{2}dz.$

$\int e^{\boxed{y}}\, d\boxed{y} = e^{y} + C$

$\Downarrow$

$\int e^{\boxed{2x}}\, d(2x) = e^{2x} + C$

$= 4\int \underline{e^{\boxed{y}}}\frac{1}{2}\, d\boxed{y} \quad + 6\int \sin z \frac{1}{2} dz.$

$= 2\underline{\int e^{y} dy} \quad + 3\int \sin z\, dz$

$\underline{\int \sin \boxed{x}\, dx = -\cos x + C}$

$\int \sin(\underline{2x+1})\, d\boxed{x}$

$\neq -\cos(2x+1) + C$

$= 2(e^{y} + C_1) + 3(-\cos z + C_2)$

$= 2e^{\boxed{y}} - 3\cos\boxed{z} + (\underbrace{2C_1 + 3C_2}_{\Downarrow})$

$= 2e^{2x} - 3\cos(2x+1) + C$ ✓

(b) $\int x\,\boxed{x}\,e^{x^2}\, dx$

$\int e^{x^2} dx$

(b) $\int 8 \underline{\boxed{x}} e^{x^2} dx$

$$\cancel{\int \boxed{e^{x^2}} dx}$$

$= 8 \int x e^{\boxed{x^2}} \underline{dx}$

$\Downarrow$ let $y = \boxed{x^2} \Rightarrow dy = 2x\,dx \Rightarrow dx = \boxed{\frac{1}{2x}} dy$

$= 8 \int \cancel{x} e^{\boxed{y}} \frac{1}{2\cancel{x}} dy$

$= 4 \int e^y dy$

$= 4(e^{\boxed{y}} + C_1)$

$= 4e^{x^2} + C$

(c) $\int \frac{x}{\sqrt{\underline{x^2+7}}} \underline{dx}$

$\Rightarrow$ let $y = \boxed{x^{②}+7} \Rightarrow dy = \boxed{2x}\,dx \Rightarrow dx = \frac{1}{\boxed{2x}} dy$.

$= \int \frac{\cancel{x}}{\sqrt{y}} \cdot \frac{1}{2\cancel{x}} dy$

$y^{-\frac{1}{2}}$

$= \frac{1}{2} \int \frac{1}{\underline{\sqrt{y}}} dy$

$= \frac{1}{2}(2y^{\frac{1}{2}} + C_1)$

$= y^{\frac{1}{2}} + \frac{1}{2}C_1 \Longrightarrow (x^2+7)^{\frac{1}{2}} + C$

2 (a) $\int \frac{x^{②}}{\quad} dx$

3.(a) $\int \frac{x^{\textcircled{2}}}{x^{\textcircled{3}}+9}\,dx$

Let $y=\boxed{x^3+9} \Rightarrow dy=3x^2dx \Rightarrow dx=\frac{1}{3x^2}dy$

$\Rightarrow = \int \frac{\cancel{x^2}}{y}\cdot\frac{1}{\cancel{3x^2}}dy$

$= \frac{1}{3}\int \frac{1}{y}\,dy$

$= \frac{1}{3}(\ln y + C_1)$

$= \frac{1}{3}\ln(x^3+9)+C$

(b) $\int \frac{\sin x}{\cos x}\,dx \quad d(\cos x)$

$\boxed{d(\cos x) = -\sin x\,dx}$

$\Rightarrow$ let $\boxed{\cos x} = y \Rightarrow dy = -\sin x\,dx \Rightarrow dx = \frac{1}{-\sin x}dy.$

$= \int \frac{\cancel{\sin x}}{y}\cdot\frac{1}{-\cancel{\sin x}}dy$

$= -\int \frac{1}{y}\,dy$

$= -(\ln y + C_1) \Rightarrow -\ln(\cos x)+C$ #.

(c) $\int \frac{3x^1+11}{x^2-x-6}\,dx.$

fractional $= \frac{\text{Polynomial}}{\text{Polynomial}}.$

① power : long-division

$= \int \frac{3x+11}{(x-3)\cdot(x+2)}\,dx$

$$=\int \frac{\quad}{(x-3)(x+2)}\,dx$$

① power : long-division

② factorization

$$= \frac{A}{x-3}+\frac{B}{x+2} = \frac{3x+11}{(x-3)(x+2)} \checkmark$$

$$\Downarrow$$

$$\frac{A(x+2)+B(x-3)}{(x-3)(x+2)}$$

$$\Downarrow$$

$$\frac{(A+B)x+(2A-3B)}{(x-3)(x+2)}$$

$$\Rightarrow \begin{cases} A+B=3 \\ 2A-3B=11 \end{cases} \Rightarrow \begin{cases} A=\boxed{4} \\ B=-1 \end{cases}$$

$$=\int \frac{4}{x-3}+\frac{-1}{x+2}\,dx$$

$$=4\int \frac{1}{x-3}\,dx-\int\frac{1}{x+2}\,dx$$

$$=4\int \frac{1}{x-3}\,d(x-3)-\int\frac{1}{x+2}\,d(x+2)$$

let $u=x-3$ $\quad v=x+2$

$$=4\ln|u|-\ln|v|+C$$

$$=4\ln|x-3|-\ln|x+2|+C \checkmark$$

(d) $\int \frac{6}{x^2-1}\,dx$

$\int \frac{\text{top}}{\text{bottom}}\,dx$

$$\Rightarrow \frac{\frac{d}{dx}(\text{bottom})}{\text{top}} \neq \text{constant.} \checkmark$$

$$=\int\frac{6}{(x-1)(x+1)}\,dx$$

$$\checkmark \quad \frac{A}{x-1}+\frac{B}{x+1}=\frac{6}{(x-1)(x+1)}$$

$$\frac{\frac{d}{dx}(x^2-1)}{6}=\frac{2x}{6}\neq \text{constant}$$

$\frac{}{x-1} + \frac{}{x+1} = \frac{}{(x-1)(x+1)}$ ✓ $\quad 6 \quad = 6 \neq$ constant.

$$\Downarrow$$

$$\frac{A(x+1)+B(x-1)}{(x-1)(x+1)}$$

$$\Downarrow$$

$$\frac{(A+B)x+(A-B)}{(x-1)(x+1)} \Rightarrow \begin{cases} A+B=0 \\ A-B=6 \end{cases} \Rightarrow \begin{cases} A=3 \\ B=-3 \end{cases}$$

$$= \int \left(\frac{3}{x-1} + \frac{-3}{x+1}\right) dx$$

$$= 3\int \frac{1}{x-1}\, d(x-1) - 3\int \frac{1}{x+1}\, d(x+1)$$

$\ln x - \ln y = \ln \frac{x}{y}$

$$= 3\ln|x-1| - 3\ln|x+1| + C$$

$$= 3\ln\left|\frac{x-1}{x+1}\right| + C.$$

(e) $\int \frac{x^{\boxed{3}}}{x^{\boxed{2}}-1}\, dx$ ① Power: greater or equal

Long-division.

$$\begin{array}{r} x \\ x^2-1 \overline{) \, x^3+0} \\ x^3-x \\ \hline x \end{array} \Rightarrow \underline{x^3 = x(x^2-1)+x}$$

$$= \int \frac{x(x^2-1)+x}{x^2-1}\, dx$$

Method I: substitution.

$$= \int \left(x + \frac{x}{x^2-1}\right) dx \quad ✓ \quad \frac{\frac{d(x^2-1)}{dx}}{x} = \frac{2x}{x} = 2!!$$

constant!

$$= \int x\,dx + \boxed{\int \frac{x}{x^2-1}\, dx}$$

Let $u = x^2-1$.

$$= \underbrace{\int x\,dx} + \left| \int \frac{x}{\underset{\Downarrow\ (x+1)(x-1)}{x^2-1}} \underline{dx} \right|$$

$$= \frac{1}{2}x^2 + \int \frac{\cancel{x}}{u} \cdot \frac{1}{2\cancel{x}} du$$

$$= \frac{1}{2}x^2 + \frac{1}{2}\int \frac{1}{u} du$$

$$= \frac{1}{2}x^2 + \frac{1}{2}\ln|u| + C$$

$$= \frac{1}{2}x^2 + \underbrace{\frac{1}{2}\ln|x^2-1|} + C.$$

Let $\underbrace{u = x^2-1}$.

$$\Rightarrow du = d(x^2-1) = 2x\,dx$$

$$\Rightarrow dx = \frac{1}{2x} du.$$

Method II: separation.

$$\int \frac{x}{(x-1)(x+1)} dx$$

$$\Downarrow$$

$$\frac{A}{x-1} + \frac{B}{x+1} = \underbrace{\frac{x}{(x-1)(x+1)}}$$

$$\Downarrow$$

$$\frac{A(x+1)+B(x-1)}{(x-1)(x+1)} \Updownarrow = \begin{cases} A+B=1 \\ A-B=0 \end{cases}$$

$$\Downarrow$$

$$\underbrace{\frac{(A+B)x+(A-B)}{(x-1)(x+1)}} \Rightarrow \begin{cases} A=\frac{1}{2} \\ B=\frac{1}{2}. \end{cases}$$

$$= \int \left(\frac{\frac{1}{2}}{x-1} + \frac{\frac{1}{2}}{x+1} \right) dx$$

$$= \frac{1}{2}\left(\int \frac{1}{x-1} dx + \int \frac{1}{1+x} dx \right)$$

$$= \frac{1}{2}\left(\ln|x-1| + \ln|x+1| \right) + C.$$

$$= \frac{1}{2}\left(\ln|x^2-1| \right) + C$$

$\ln x + \ln y$

$= \ln(xy)$

(g) $\int \frac{x+2}{x^2+1} dx.$ ① power ① factorization

$$= \int \frac{x}{x^2+1} \underline{\underline{dx}} + 2\int \frac{1}{x^2+1} dx$$

$$\int \frac{1}{1+x^2} dx = \underline{\underline{\tan^{-1}(x) + C}}.$$

$$\frac{\frac{d(x^2+1)}{dx}}{x} = \frac{2x}{x} = 2!!!$$

$$\int \frac{1}{x^2+a^2} dx = \frac{1}{}\tan^{-1}(x).$$

$$\frac{dx}{x} = \frac{2x}{x} = 2 \;!!!$$

constant.

let $u = x^2 + 1$.

$du = d(x^2+1) = 2x\,dx$

$\Rightarrow dx = \frac{1}{2x}\,du$.

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$$

$$= \int \frac{\frac{1}{a^2}}{\underbrace{\left(\frac{x}{a}\right)^2}_{u}+1}\,dx$$

$$= \frac{1}{a}\int \frac{1}{\left(\frac{x}{a}\right)^2+1}\,d\left(\frac{x}{a}\right)$$

$$= \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$$

$$= \int \frac{\cancel{x}}{u}\cdot\frac{1}{2\cancel{x}}\,du + 2\tan^{-1}(x) + C.$$

$$= \frac{1}{2}\int \frac{1}{u}\,du + 2\tan^{-1}(x) + C$$

$$= \frac{1}{2}\ln|x^2+1| + 2\tan^{-1}(x) + C \quad \#$$

(h) $\int \frac{x+1}{x^2+6x+10}\,dx$

① ~~long division~~

② ~~factorization~~

③ check ~~substitution~~

④ complete square

$$\frac{\frac{d(x^2+6x+10)}{dx}}{x+1} = \frac{2x+6}{x+1} \neq \text{constant}$$

$$= \int \frac{x+1}{(x^2+6x+9)+1}\,dx$$

$$= \int \frac{x+1}{(x+3)^2+1}\,dx$$

$$= \int \frac{x+3-2}{(x+3)^2+1}\,dx$$

$$= \int \frac{x+3}{(x+3)^2+1}\,dx - \underbrace{2\int \frac{1}{(x+3)^2+1}\,dx}$$

$\Downarrow$

$$2\int \frac{1}{(x+3)^2+1}\,d(x+3)$$

$\Downarrow$ let $u = x+3$

$$\frac{\frac{d((x+3)^2+1)}{dx}}{x+3} = \frac{2(x+3)}{x+3} = 2 \;!!$$

Constant

$$x+3 \qquad - \qquad x+3 = 2!!$$

$$\underbrace{(x+3)^2+\underline{1}} = v \qquad \text{Constant.}$$

$$dv = d((x+3)^2+1)$$

$$= 2(x+3)\,dx$$

$$\Rightarrow dx = \frac{1}{2(x+3)}\,dv$$

$$\Rightarrow \int \frac{\cancel{x+3}}{v} \cdot \frac{1}{2\cancel{(x+3)}}\,dv.$$

$$= \frac{1}{2}\int \frac{1}{v}\,dv$$

$$= \frac{1}{2}\ln|(x+3)^2+1| + C \quad Ⓐ$$

$\Downarrow$ let $\underline{u=x+3}$

$$2\int \frac{1}{u^2+1}\,du.$$

$$\Downarrow$$

$$2\tan^{-1}(x+3)+C \quad Ⓑ$$

Combine Ⓐ and Ⓑ

$$= \frac{1}{2}\ln|(x+3)^2+1| - 2\tan^{-1}(x+3)+C$$

#.

P4: Integration by part.

$$\underline{\int \underline{f}\,\underline{g}\,d\underline{x}}$$

$$\boxed{\int f\,dx = F(x)}$$

$$F'(x) = f(x)$$

$$= \int g\,dF(x)$$

$$= g(x)\cdot F(x) - \int F(x)\,dg(x)$$

$$= g(x)\cdot F(x) - \underline{\int F(x)\cdot \underline{g'(x)}\,dx}$$

✓

(a) $\displaystyle\int \overset{f}{\underline{x}} \cdot \overset{g}{\underline{\ln x}}\,dx$

$$= \int \ln x\, d(\tfrac{1}{2}x^2)$$

$$= \frac{1}{2}x^2 \cdot \ln x - \int \frac{1}{2}x^2\, \underline{d(\ln x)}$$

$$= \frac{1}{2}x^2 \cdot \ln x - \int \frac{1}{2}x^{\cancel{2}} \cdot \frac{1}{\cancel{x}}\,dx$$

$$= \frac{1}{2}x^2\ln x - \frac{1}{2}\underline{\int x\,dx}$$

$$= \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + \underline{C}$$

(c) $\displaystyle\int \overset{g}{\underline{x^2}} \cdot \overset{f}{\underline{e^x}}\,dx$

(c) $\int \underline{x^2} \cdot \underline{e^x}\, dx$

$= \int x^2 d(e^x)$

$= x^2 \cdot e^x - \int e^x d(x^2)$ } 1^{st}-time

$= x^2 \cdot e^x - \int \underline{e^x\, 2x^1}\, dx$

$= x^2 \cdot e^x - \int 2x\, d(e^x)$ } 2^{nd}-time

$= x^2 \cdot e^x - [2x \cdot e^x - \int e^x d(2x)]$

$= x^2 \cdot e^x - 2x \cdot e^x + 2\int e^x dx$

$= x^2 \cdot e^x - 2x \cdot e^x + 2e^x + C$

d) $\boxed{\int \underline{\sin x} \cdot \overset{f}{\underline{e^x}}\, dx} \Rightarrow I$

$I = \sin x\, e^x - \cos x\, e^x - I$

$\Rightarrow 2I = \sin x \cdot e^x - \cos x \cdot e^x$

$\Rightarrow I = \dfrac{\sin x \cdot e^x - \cos x \cdot e^x}{2} + C$

$= \int \sin x\, d(e^x)$

$= \sin x \cdot e^x - \int e^x d(\sin x)$ } 1^{st}-time

$= \sin x \cdot e^x - \int e^x \cdot \underline{\cos x}\, dx$

$= \sin x \cdot e^x - \int \cos x\, d(e^x)$ } 2^{nd}-time

$= \sin x \cdot e^x - [\cos x \cdot e^x - \int e^x d(\cos x)]$

$= \sin x \cdot e^x - \cos x \cdot e^x - \underset{\Downarrow I}{\boxed{\int e^x \cdot \sin x\, dx}} + C$

(f) $\int \overset{g}{\underline{\tan^{-1}(x)}} \cdot \overset{f}{1}\, dx$

$\int \left(\dfrac{1}{x^2+1}\right) dx = \tan^{-1}(x) + C$

[illegible]

$$\int \Big(\ x^2+1\Big) dx = \tan^{-1}(x)+C$$

$$= \int \tan^{-1}(x)\, d(x)$$

$$= \tan^{-1}(x)\cdot x - \int x \, d(\tan^{-1}(x))$$

① integration by part

$$= \tan^{-1}(x)\cdot x - \int x \cdot \frac{1}{x^2+1}\, dx$$

$$\frac{\frac{d(\text{bottom})}{dx}}{\text{top}} \overset{?}{=} \text{constant}$$

let $u = x^2+1.$

$\Rightarrow du = 2x \cdot dx$

$\Rightarrow dx = \frac{1}{2x} du.$

$$\Rightarrow \frac{\frac{d(x^2+1)}{dx}}{x} = \frac{2x}{x} = 2!$$

② substitution.

$$= \tan^{-1}(x)\cdot x - \int \frac{\cancel{x}}{u}\cdot\frac{1}{2\cancel{x}}\, du.$$

$$= \tan^{-1}(x)\cdot x - \frac{1}{2}\ln|\underset{1+x^2}{u}| + C$$

$$= \tan^{-1}(x)\cdot x - \frac{1}{2}\ln|x^2+1| + C \quad ✓$$

P5: Definite integral:

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b$$

$$= F(b) - F(a).$$

$$\int_0^2 8x^3\,dx$$

$$= 8\int_0^2 x^3\,dx$$

$$= \left(8\frac{x^4}{4}+C\right)\Big|_0^2 = \left(8\cdot\frac{2^4}{4}+C\right) - \left(8\cdot\frac{0^4}{4}+C\right)$$

$$= 32$$

(b) $\int_2^3 y e^y\, dy = (ye^y - e^y + C)\Big|^3$

(b) $\int_2^3 y e^y \, dy = (ye^y - e^y + C)\Big|_2^3$

$\int y e^y dy$

$= \int y \, d(e^y)$

$= y \cdot e^y - \int e^y dy$

$= y \cdot e^y - e^y + C$

$= 3e^3 - e^3 + C - (2e^2 - e^2 + C)$

$= 2e^3 - e^2$ #

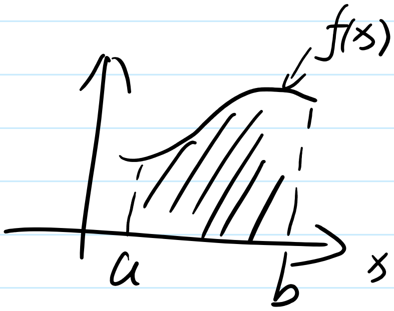
(c) $\int_1^2 x \ln x \, dx$. refer to P4 (a)

$= \left(\frac{1}{2} x^2 \ln x - \frac{1}{4} x^2 + C\right)\Big|_1^2$

$= \frac{1}{2} \times 4 \times \ln 2 - 1 + \frac{1}{4}$

$= 2\ln 2 - \frac{3}{4}$

P6: geometric meaning of integration.

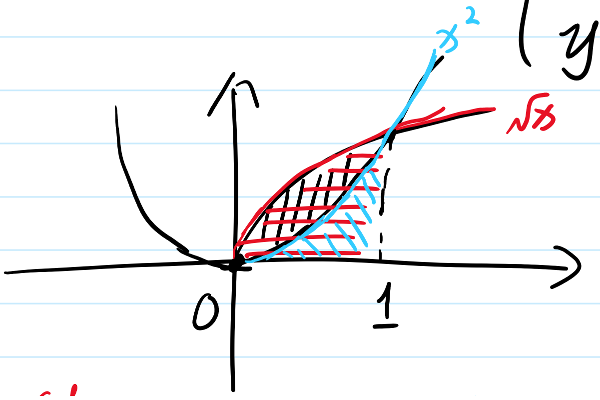


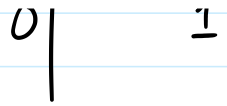
$S = \int_a^b f(x) dx$

(a) area enclosed by $\begin{cases} y = x^2 \\ y = \sqrt{x}. \end{cases}$ $\Rightarrow x^2 = \sqrt{x}$

$\Rightarrow x_1 = 0$

$x_2 = 1.$





$$S = \int_0^1 \sqrt{x}\,dx - \int_0^1 x^2\,dx.$$

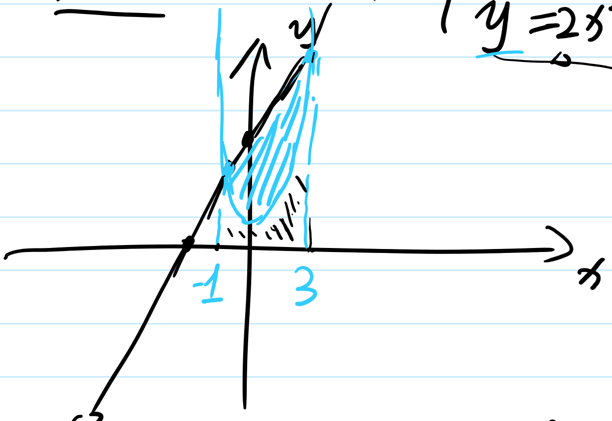
$$= \frac{2}{3}x^{\frac{3}{2}}\Big|_0^1 - \frac{1}{3}x^3\Big|_0^1$$

$$= \frac{2}{3} - \frac{1}{3} = \frac{1}{3}.$$ ✓

(b) Area enclosed $\begin{cases} y=4x+16 \checkmark \\ y=2x^2+10 \checkmark \end{cases} \Rightarrow 4x+16=2x^2+10$

$\Rightarrow x^2-2x-3=0$

$\Rightarrow x_1=3,\ x_2=-1.$



✓

$$\int_{-1}^{3}(4x+16)\,dx - \int_{-1}^{3}(2x^2+10)\,dx$$

$$= \int_{-1}^{3}(-2x^2+4x+6)\,dx$$

$$= \left(-2\frac{x^3}{3}\right)\Big|_{-1}^{3} + \left(4\frac{x^2}{2}\right)\Big|_{-1}^{3} + (6x)\Big|_{-1}^{3}$$

$$= \boxed{\frac{64}{3}}$$ #.

Chapter 4

Monday, April 20, 2020 3:55 PM

MA0101 Tutorial Class (TB) session.

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Chapter 4: Complex number.

$\underbrace{a}_{\text{Real}} + \underbrace{bi}_{\text{Im}}$ ✓

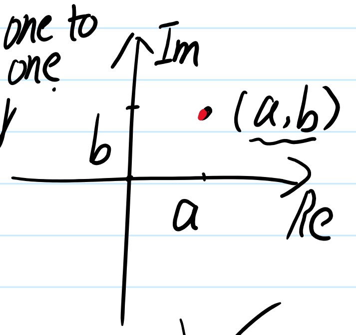
Real number: rational number: 1, 2, 3, ⋯ $\frac{2}{3}, \frac{1}{2}, \cdots$

irrational number: $e, \pi, \sqrt{2}$.

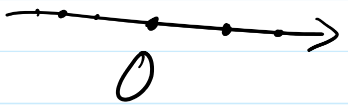
$x^2 = -1$, $\Rightarrow x = \pm\sqrt{-1} = \pm i$ $\boxed{i^2 = -1}$ $x^2 = 2 \Rightarrow x = \pm\sqrt{2}$

Complex unit.

one to one

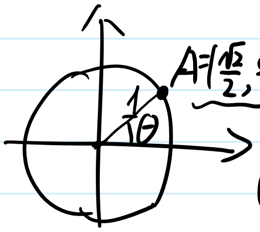


Argand Diagram ✓



✓

Representation: Polar form. ① radius ② argument.



$A = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ (A): $r = 1, \theta = \frac{\pi}{4}$.

✓ A: $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$.

$(r, \theta) = (1, \frac{\pi}{4})$

✓ $re^{i\theta} = r(\cos\theta + i\sin\theta)$

$= r(\cos\theta + i\sin\theta)$

$$\boxed{\text{Euler formula: } e^{i\theta} = \cos\theta + i\sin\theta}$$

A: $1e^{i\frac{\pi}{4}} = 1\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$
$= \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$

e i π: $e^{i\pi} = \cos\pi + i\sin\pi$
$= -1$

1. (a) $(7+3i)+(3-6i)$
$= (7+3) + (3-6)i$
$= 10 - 3i.$

Addition / subtraction

(b) $(2+i)-(3-2i)$
$= (2-3)+(i+2i)$
$= -1 + 3i.$

(c) $(1+3i)\cdot(4-i)$
$= 4 + 12i - i - 3\overset{-1}{i^2}$
$= 4 + 11i + 3$
$= 7 + 11i.$

Multiplication
$(x+y)\cdot(a+b)$
$= xa + xb + ya + yb.$ ✓

(e) $\dfrac{1+3i}{(4-i)}$ ✓

$4-i$
$\Updownarrow$
$4+i$

Division:

$$\boxed{\text{complex conjugate}}$$

$a+bi.$ ✓
$\Updownarrow$
$a-bi$ ✓

$= \dfrac{(1+3i)\cdot(4+i)}{(4-i)\cdot(4+i)}$

$= \dfrac{4+12i+i+3\overset{-1}{i^2}}{16-\cancel{4i}+\cancel{4i}-\underset{-1}{i^2}} = \dfrac{1+13i}{17}$ ✓ $= \underbrace{\left(\frac{1}{17}+\frac{13}{17}i\right)}_{a\ +\ bi}$

✓ $(a+bi)\cdot(a-bi)$
$= a^2 + \cancel{abi} - \cancel{abi} - b^2\underset{-1}{i^2}$

$\Downarrow$ -1 ✓ $\underline{a + bi}$ ✓

$= a^2 + \cancel{abi} - \cancel{abi} - b^2 \underset{-1}{i^2}$

$= a^2 + b^2.$

(f) $\frac{1+3i}{i}$

"i" $\Downarrow$ $0+1i \Leftrightarrow 0-1i = \boxed{-i}$

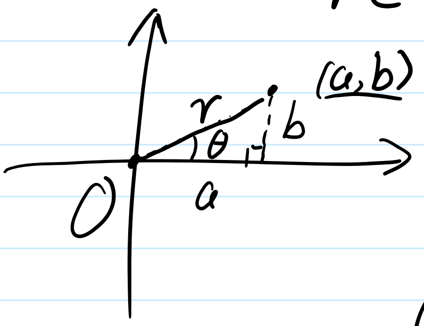
$= \frac{(1+3i)\cdot(-i)}{i\cdot(-i)}$

$= \frac{-i-3i^2}{-\underset{-1}{i^2}} = \frac{3-i}{1} = \underline{\underline{3-i}}$

2、Ordinary form $\Leftrightarrow$ Polar form.

$a+bi$

$$re^{i\theta} = r(\cos\theta + i\sin\theta)$$



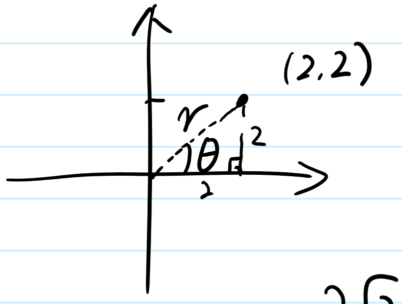
Argand diagram

$r = \sqrt{a^2+b^2}$

$\underline{\underline{\theta}} \Rightarrow \tan\theta = \frac{b}{a}.$

$\Rightarrow \theta = \tan^{-1}\left(\frac{b}{a}\right).$

(a) $\underline{\underline{2+2i}}$



$r = \sqrt{2^2+2^2} = \sqrt{8} = 2\sqrt{2}.$

$\theta \Rightarrow \tan\theta = \frac{2}{2} = 1$

$\Rightarrow \theta = \tan^{-1}(1) = \frac{\pi}{4}.$ ✓

$\theta \pm 2k\pi,$

$k = 1, 2, 3,$

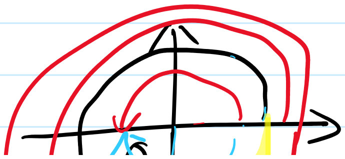
$2\sqrt{2}e^{i\frac{\pi}{4}} = 2\sqrt{2}\left(\cos\left(\frac{\pi}{4} \pm 2k\pi\right) + i\sin\left(\frac{\pi}{4} \pm 2k\pi\right)\right)$ ✓ $\boxed{2\pi}$

(b) $-7i. = 0-7i$

✓ Principal range of $\boxed{\theta}$.

$(0,-7)$

$r = \sqrt{0^2+(-7)^2} = 7.$



$\theta = -\pi$

$\underline{\theta} = -\frac{\pi}{2}$

$[0,\pi) \cup [-\pi,0)$

$(-\pi,\pi]$

$\frac{3\pi}{2} - 2\pi$

$$re^{i\theta} = 7e^{i(-\frac{\pi}{2})}$$

$$= 7\left[\cos(-\tfrac{\pi}{2}) + \sin(-\tfrac{\pi}{2})i\right] = -\frac{\pi}{2}.$$

$\frac{3\pi}{2}$ $\frac{3\pi}{2}$ ✓

$[-\pi \pm 2k\pi, \pi \pm 2k\pi) \longrightarrow (-\pi,\pi]$

$\frac{3\pi}{2}$ ✗

$-\frac{\pi}{2}$ ✓

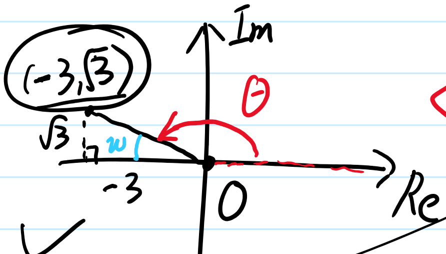
$\frac{3\pi}{2} + 2\pi$

$+2k\pi$

$\frac{3\pi}{2} + 4\pi$

⋮

2. (c) $-3 + i\sqrt{3}$ ✓



✓

$$r = \sqrt{(-3)^2 + (\sqrt{3})^2} = \sqrt{12} = 2\sqrt{3}.$$

$\boxed{\theta + w = \pi}$

$\boxed{\theta} = \pi - w = \pi - \frac{\pi}{6} = \frac{5\pi}{6}. > \frac{\pi}{2}$

$w = \tan^{-1}(\frac{\sqrt{3}}{3}) = \frac{\pi}{6} < \frac{\pi}{2}$

$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{-3}\right) = \frac{5\pi}{6}$ ||

$$re^{i\theta} = 2\sqrt{3}e^{i\frac{5\pi}{6}} \underset{\text{Euler formula.}}{=} 2\sqrt{3}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right).$$

(d) $\frac{2+2i}{-3+i\sqrt{3}}$

① by 1(e), Division ⇒ Ordinary ⇒ Polar form. ✓

① by 1(e), Division ⇒ Ordinary ⇒ Polar form. ✓

② ordinary ⇒ Polar form ⇒ division

↓ by 2(a),(c)

$$= \frac{2\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)}{2\sqrt{3}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)}$$

Euler formula ✓

$$= \frac{\cancel{2}\sqrt{2}\, e^{i\frac{\pi}{4}}}{\cancel{2}\sqrt{3}\, e^{i\frac{5\pi}{6}}} \quad ✓$$

$$= \frac{\sqrt{2}}{\sqrt{3}} e^{\left(i\frac{\pi}{4} - i\frac{5\pi}{6}\right)} = \frac{\sqrt{2}}{\sqrt{3}} e^{i\left(-\frac{7\pi}{12}\right)} = \frac{\sqrt{6}}{3}\left(\cos\left(-\frac{7\pi}{12}\right) + i\sin\left(-\frac{7\pi}{12}\right)\right)$$

Euler formula.

$$\frac{e^x}{e^y} = e^{x-y} \quad ✓$$

$$e^x \cdot e^y = e^{x+y}$$

(e) $(2+2i)^8 = \underbrace{(2+2i)\cdot(2+2i)\cdots(2+2i)}_{“8”}$

by (a)

$$= \left(2\sqrt{2}\cdot\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right)^8$$

⇓ Euler

$$= (2\sqrt{2})^8 \cdot \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^8$$

$$= 2^{12}\cdot\left(e^{i\frac{\pi}{4}}\right)^8 = 2^{12}\cdot\left(e^{i\frac{\pi}{4}\cdot 8}\right) = 2^{12}\left(e^{i2\pi}\right)$$

$$= 2^{12}(\cos 2\pi + i\sin 2\pi) \quad \#$$

$$(e^x)^y = e^{x\cdot y}$$

$\cos 8\cdot\frac{\pi}{4} + i\sin 8\cdot\frac{\pi}{4} = \cos 2\pi + i\sin 2\pi.$

3、De Moivre's theorem:

$(\cos x + i\sin x)^n = \cos nx + i\sin nx$ ✓

Proof: $(e^{ix})^n = e^{inx} \underset{\text{Euler}}{=} (\cos nx + i\sin nx)$

① usage : De Moivre

Euler

① way : De Moivre.

$(\cos x + i\sin x)^3 = \underbrace{\cos 3x}_{\text{real}} + i\sin 3x$

② way : Expansion, binomial theorem: "i" $i^2=-1$

$= \binom{3}{0}\overset{(\cos x)^3}{\cos^3 x} + \binom{3}{1}\overset{(\cos x)^2\cdot(i\sin x)}{\cos^2 x(i\sin x)} + \binom{3}{2}\cos x(i\sin x)^2 + \binom{3}{3}(i\sin x)^3$

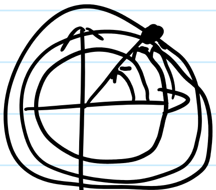
$= \underbrace{\cos^3 x - 3\cos x\sin^2 x}_{\text{real}}$

$+ i(3\cos^2 x\sin x - \sin^3 x)$

① identity: $\cos 3x = \cos^3 x - 3\cos x\sin^2 x$

② identity: $\sin 3x = 3\cos^2 x\sin x - \sin^3 x.$

$$\boxed{\begin{matrix}\cos(x^3) \neq \cos^3 x = (\cos x)^3 \\ \not\parallel \\ \cos 3x\end{matrix}}$$

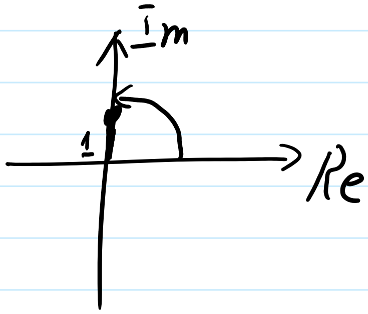


$+2k\pi$, $k = \cdots, -2, -1, 0, 1, 2, \cdots$

$(z)^{\frac{1}{2}}$ / $(z)^{\frac{1}{5}}$ $(e^x)^{\frac{1}{2}} = e^{\frac{x}{2}}$

two distinct solutions

4. (a) $z^{②} = i \xrightarrow{\text{formally}} z = \sqrt{i} = (i)^{\frac{1}{2}}$?



$r = 1$

$\theta = \frac{\pi}{2}$

$i = re^{i\theta} = 1\cdot e^{i\frac{\pi}{2}}$

$z = (1\cdot e^{i(\frac{\pi}{2}+2k\pi)})^{\frac{1}{2}}$ $k = \cdots, -2, -1, 0, 1, 2, \cdots$

$= e^{i\frac{\frac{\pi}{2}+2k\pi}{2}}$

$= e^{i(\frac{\pi}{4}+k\pi)}$

Principle range

Principle range

$\underline{(-\pi,\pi]}$

$$=e^{i(\frac{\pi}{4}+k\pi)}$$

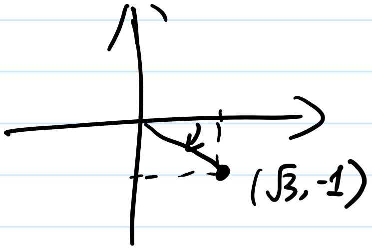
$$\begin{cases} k=0, & \underline{\underline{e^{i\frac{\pi}{4}}}} = \overset{k=2}{e^{i(\frac{\pi}{4}+2\pi)}} = \cdots\cdots \\ k=-1, & \underline{\underline{e^{i(-\frac{3\pi}{4})}}} = \underset{k=1}{e^{i\left(\boxed{\frac{5\pi}{4}}\right)}} = \cdots\cdots \end{cases}$$

Repeatedly appear

The solutions are $e^{i\frac{\pi}{4}}$ and $e^{i(-\frac{3\pi}{4})}$ ✓

five distinct solution

(d) $z^{\boxed{5}} = \sqrt{3}-i \Longrightarrow z = (\underline{\underline{\sqrt{3}-i}})^{\boxed{\frac{1}{5}}}$



$$r=\sqrt{(\sqrt{3})^2+(-1)^2}=2$$

$$\theta=\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)=-\frac{\pi}{6}$$

$$\sqrt{3}-i=re^{i\theta}=\underline{2e^{i(-\frac{\pi}{6})}}$$

$$z=(2e^{i(-\frac{\pi}{6}+2k\pi)})^{\boxed{\frac{1}{5}}} = 2^{\frac{1}{5}}\cdot e^{i\frac{-\frac{\pi}{6}+2k\pi}{5}} \quad k=\cdots,-2,-1,0,1,2,\cdots$$

$$=2^{\frac{1}{5}}\cdot e^{i\underline{\underline{(-\frac{\pi}{30}+\frac{2}{5}k\pi)}}}$$

"Principle range:"

$[-\pi,\pi]$

$$\begin{cases} k=0, & 2^{\frac{1}{5}}e^{i(-\frac{\pi}{30})} \\ k=1, & 2^{\frac{1}{5}}e^{i(\frac{11\pi}{30})} \\ k=2, & 2^{\frac{1}{5}}e^{i(\frac{23\pi}{30})} \end{cases} \qquad \begin{cases} k=-1, & 2^{\frac{1}{5}}e^{i(-\frac{13\pi}{30})} \\ k=-2, & 2^{\frac{1}{5}}e^{i(-\frac{5\pi}{6})} \end{cases}$$

~~k=3~~ ⋮ ~~k=-3~~ ⋮