Overview of the Mini-Course at CUHK (2021/22 Term 1): Introduction of the Solution to Spatially Homogeneous Boltzmann Equation as a Probability Measure

Kunlun Qi (kunlun.qi@cuhk.edu.hk)

Abstract

In this mini-course, the development of the spatially homogeneous theory to the Boltzmann equation will be briefly introduced, especially for the well-posedness result of the Cauchy problem in the space of probability measure defined via the Fourier transform. Besides the original solution with finite energy, the infinite energy case is not a priori excluded from the consideration either, where the Bobylev identity and Fourier-based probability metric will be discussed as two powerful tools.

Rough Arrangement:

So far, a rough arrangement of the four lectures is provided as following, where some adjustments might happen according to the actual progress:

(I) In the lecture1 (3:30 pm - 5:30 pm, Oct. 4), the Fourier transformation in the kinetic equation and its induced probability metric will firstly introduced, including their basic calculus rules and contractive property, which then leads to the uniqueness of the solution to homogeneous Boltzmann equation with finite energy.

(II) In the lecture2 (3:30 pm - 5:30 pm, Oct. 11) - lecture3 (3:30 pm - 5:30 pm, Oct. 18), the well-posedness result of the homogeneous Boltzmann equation will be derived in the measure valued sense from cutoff to non-cutoff kernel, where the infinite energy solutions are also not a priori excluded from the consideration; moreover, the asymptotic behaviour towards the self-similar profile will also partially discussed.

(III) In the lecture4 (3:30 pm - 5:30 pm, Oct. 25), the contents mentioned above shall firstly be finished off, then, compared with the Maxwellian molecule, the similar methodology will be illustrated how to solve the general hard/soft potential case in the probability measure space; if time permits, some further applications to other kinetic-related model, e.g., dissipative inelastic Boltzmann equation, will be discussed as well.

Corresponding and Relevant Materials

The following materials are, in chronological order, referred to the development of the study about the solution to spatially homogeneous Boltzmann equation as a probability measure, where the Fourier Transformation plays a critical role.

KQ: This list is not intended to be completely covered in the mini-course, which is definitely impossible, but to partly reflect the history and hopefully present a big picture about how the research of the homogeneous Boltzmann equation in probability measure sense developed: from cutoff to non-cutoff, from the Maxwellian molecule to hard/soft potential, from higher-order moments requirement to lower-order... The selection is biased in favor of personal taste.

[1] A. V. Bobylev. A class of invariant solutions of the Boltzmann equation. *Dokl. Akad. Nauk SSSR*, 231(3):571–574, 1976.

[2] A. V. Bobylev. The theory of the nonlinear spatially uniform Boltzmann equation for Maxwell molecules. *Soviet Sci. Rev. Sect. C: Math. Phys. Rev., 7*, 111–233, 1988.

[3] G. Gabetta, G. Toscani, and B. Wennberg. Metrics for probability distributions and the trend to equilibrium for solutions of the Boltzmann equation. *J. Stat. Phys.*, 81(5-6):901–934, 1995.

[4] E. A. Carlen, E. Gabetta, and G. Toscani. Propagation of smoothness and the rate of exponential convergence to equilibrium for a spatially homogeneous Maxwellian gas. *Comm. Math. Phys.*, 199(3):521–546, 1999.

[5] G. Toscani and C. Villani. Probability metrics and uniqueness of the solution to the Boltzmann equation for a Maxwell gas. *J. Stat. Phys.*, 94(3-4):619–637, 1999.

[6] A. V. Bobylev and C. Cercignani. Self-similar solutions of the Boltzmann equation and their applications. *J. Stat. Phys.*, 106(5-6):1039–1071, 2002.

[7] C. Villani. A review of mathematical topics in collisional kinetic theory. In S. Friedlander and D. Serre, editors, *Handbook of Mathematical Fluid Mechanics*, volume I, pages 71–305. North-Holland, 2002.

[8] L. Desvillettes. About the use of the Fourier transform for the Boltzmann equation. volume 2*, pages 1–99. 2003. Summer School on “Methods and Models of Kinetic Theory” (M&MKT 2002).

[9] C. Villani. Mathematics of granular materials. *J. Stat. Phys.*, 124(2-4):781–822, 2006.

[10] J. A. Carrillo, G. Toscani Contractive probability metrics and asymptotic behavior of dissipative kinetic equations. *Riv. Mat. Univ. Parma*, 7(6):75–198, 2007.

[11] A. V. Bobylev, C. Cercignani, and I. M. Gamba. On the self-similar asymptotics for generalized nonlinear kinetic Maxwell models. *Comm. Math. Phys.*, 291(3):599–644, 2009.

[12] M. Cannone and G. Karch. Infinite energy solutions to the homogeneous Boltzmann equation. *Comm. Pure Appl. Math.*, 63(6):747–778, 2010.

[13] Y. Morimoto. A remark on Cannone-Karch solutions to the homogeneous Boltzmann equation for Maxwellian molecules. *Kinet. Relat. Models*, 5(3):551–561, 2012.

[14] M. Cannone and G. Karch. On self-similar solutions to the homogeneous Boltzmann equation. *Kinet. Relat. Models*, 6(4):801–808, 2013.

[15] Y. Morimoto, S. Wang, and T. Yang. A new characterization and global regularity of infinite energy solutions to the homogeneous Boltzmann equation. *J. Math. Pures Appl. (9)*, 103(3):809–829, 2015.

[16] Y. Morimoto, S. Wang, and T. Yang. Measure valued solutions to the spatially homogeneous Boltzmann equation without angular cutoff. *J. Stat. Phys.*, 165(5):866–906, 2016.

[17] Y. Morimoto, T. Yang, and H. Zhao. Convergence to self-similar solutions for the homogeneous Boltzmann equation. *J. Eur. Math. Soc.*, 19(8):2241–2267, 2017.

[18] K. Qi. Measure Valued Solution to the Spatially Homogeneous Boltzmann Equation with Inelastic Long-Range Interactions. *Preprint*, 2020.

[19] A. V. Bobylev, A. Nota, J. J. L. Velázquez. Self-similar asymptotics for a modified Maxwell-Boltzmann equation in systems subject to deformations. *Comm. Math. Phys.*, 380(1):409–448, 2020.

[20] K. Qi. On the measure valued solution to the inelastic Boltzmann equation with soft potentials. *J. Stat. Phys.*, 183(27), 2021.

Preliminary Knowledges:

KQ: The following elementary introductions about the Boltzmann equation, especially the collision operator, are priorly assumed to be familiar with our audience, which would be no longer over-repeated during the mini-course.

0.1 The Spatially Homogeneous Boltzmann equation.

In the spatially homogeneous theory of the Boltzmann equation, one is interested in the solution $f(t,x,v)$ which does not depend on the x space variable. This view of point is pretty common in physics, especially when it comes to the problems focusing on the collision operator, as the collision integral operator only acts on the velocity dependence. On the other hand, the interests towards the spatially homogeneous study also arise from the numerical analysis, since almost all numerical schemes succeed from the splitting of the transport step and collision step.

In this case, the homogeneous Boltzmann equation in $\mathbb{R}^3$ reads:

$$\partial_t f(t,v) = Q(f,f)(t,v), \tag{0.1}$$

with the non-negative initial condition,

$$f(0,v) = F_0(v), \tag{0.2}$$

where the unknown $f = f(t,v)$ is regarded as the density function of a probability distribution, or more generally, a probability measure; and the initial datum F_0 is also assumed to be a non-negative probability measure on $\mathbb{R}^3$.

The right hand side of (0.1) is the so-called Boltzmann collision operator,

$$\begin{aligned} Q(f,f)(v) &= \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B_\sigma(v-v_*,\sigma)\,[f(v')f(v'_*) - f(v)f(v_*)]\,\mathrm{d}\sigma\,\mathrm{d}v_* \\ &= \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B_\omega(v-v_*,\sigma)\,[f(v')f(v'_*) - f(v)f(v_*)]\,\mathrm{d}\omega\,\mathrm{d}v_*, \end{aligned} \tag{0.3}$$

where (v', v'_*) and (v, v_*) represent the velocity pairs before and after a collision, which satisfy the conservation of momentum and energy:

$$v' + v'_* = v + v_*, \quad |v'|^2 + |v'_*|^2 = |v|^2 + |v_*|^2, \tag{0.4}$$

so that (v', v'_*) can be expressed in terms of (v, v_*) as

$$\begin{aligned} v' &= \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2}\sigma, \\ \text{or} \quad v' &= v - v[(v - v_* \cdot \omega)]\omega, \quad v'_* = v + v[(v - v_* \cdot \omega)]\omega, \end{aligned} \tag{0.5}$$

where both of σ and ω are a vector varying over the unit sphere $\mathbb{S}^2$. And this also easily implies

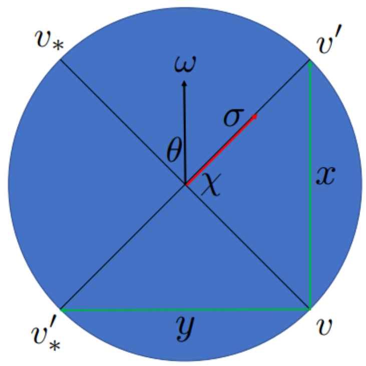

Figure 1: Velocity and unit vector during a classical elastic collision.

the relations

$$v \cdot v_* = v' \cdot v'_*, \quad |v - v_*| = |v' - v'_*|, \quad (v - v_*) \cdot \omega = -(v' - v'_*) \cdot \omega. \tag{0.6}$$

0.2 The Boltzmann collision kernel.

The collision kernel B is a non-negative function that depends only on $|v - v_*|$ and cosine of the deviation angle θ, whose specific form can be determined from the intermolecular potential using classical scattering theory. For example, in the case of inverse power law potentials $U(r) = r^{-(\mathrm{s}-1)}, 2 < \mathrm{s} < \infty$, where r is the distance between two interacting particles, B can be separated as the kinetic part and angular part:

$$B(v - v_*, \sigma) = B(|v - v_*|, \cos\theta) = b(\cos\theta)\Phi(|v - v_*|), \quad \cos\theta = \frac{\sigma \cdot (v - v_*)}{|v - v_*|}, \tag{0.7}$$

where kinetic collision part $\Phi(|v - v_*|) = |v - v_*|^\gamma$, $\gamma = \frac{\mathrm{s}-5}{\mathrm{s}-1}$, includes hard potential ($\gamma > 0$), Maxwellian molecule ($\gamma = 0$) and soft potential ($\gamma < 0$). The kernel (0.7) encompasses a wide range of potentials, among which we mention two extreme cases [7]:

(i) $\mathrm{s} = \infty$, $\gamma = 1$, $\nu = 0$ corresponds to the hard spheres, where B is only proportional to $|v - v_*|$,

$$B(|v - v_*|, \cos\theta) = K|v - v_*|, \quad K > 0; \tag{0.8}$$

(ii) s = 2, $\gamma = -3$, $\nu = 2$ corresponds to the Coulomb interaction, where B is given by the famous Rutherford formula,

$$B(|v - v_*|, \cos\theta) = \frac{1}{|v - v_*|^3 \sin^4(\theta/2)}. \tag{0.9}$$

Besides, another theoretical collision kernel that could lead to many explicit calculations is the well-known *Maxwellian molecules* $B(|v - v_*|, \sigma) = b\left(\frac{v-v_*}{|v-v_*|} \cdot \sigma\right) = b(\cos\theta)$, which implies that B does not depend on $|v - v_*|$. The range of deviation angle θ, namely the angle between pre- and post-collisional velocities, is a full interval $[0, \pi]$, but it is customary to restrict it to $[0, \pi/2]$ mathematically, replacing $b(\cos\theta)$ by its "symmetrized" version [16]:

$$\left[b(\cos\theta) + b(\cos(\pi - \theta))\right] \mathbf{1}_{0 \le \theta \le \frac{\pi}{2}}, \tag{0.10}$$

which amounts more or less to forbidding the exchange of particles.

0.3 Cutoff VS Non-cutoff

As it has been long known, the main difficulty in establishing the well-posedness result for Boltzmann equation is that the singularity of the collision kernel b is not locally integrable in $\sigma \in \mathbb{S}^2$. To avoid this, H. Grad gave the integrable assumption on the collision kernel b_c by a "cutoff " near singularity:

$$\int_{\mathbb{S}^2} b_c\left(\frac{v - v_*}{|v - v_*|} \cdot \sigma\right) \mathrm{d}\sigma = 2\pi \int_0^{\frac{\pi}{2}} b_c(\cos\theta) \sin\theta \, \mathrm{d}\theta < \infty. \tag{0.11}$$

However, the full singularity condition for the collision kernel with *non-cutoff assumption* is implicitly defined for the angular collision part $b(\cos\theta)$, which asymptotically behaves as $\theta \to 0^+$,

$$\sin\theta b(\cos\theta)\big|_{\theta \to 0^+} \sim K\theta^{-1-\nu}, \quad \nu = \frac{2}{s-1}, \quad 0 < \nu < 2 \quad \text{and} \quad K > 0, \tag{0.12}$$

or in "symmetrized" manner,

$$\exists \alpha_0 \in (0, 2], \quad \text{such that} \quad \int_0^{\frac{\pi}{2}} \sin^{\alpha_0}\left(\frac{\theta}{2}\right) b(\cos\theta) \sin\theta \mathrm{d}\theta < \infty, \tag{0.13}$$

which can handle the strongly singular kernel b in (0.12) with some $0 < \nu < 2$ and $\alpha_0 \in (\nu, 2]$. Besides, we further illustrate that the non-cutoff assumption (0.13) can be rewritten as

$$(1 - s)^{\frac{\alpha_0}{2}} b(s) \in L^1[0, 1), \quad \text{for } \alpha_0 \in (0, 2], \tag{0.14}$$

by means of the transformation of variable $s = \cos\theta$ in the symmetric version of b. As mentioned in [13, Remark 1], the full *non-cutoff assumption* (0.13), or equivalently (0.14), is the extension of the mild non-cutoff assumption of the collision kernel b used in [12], namely,

$$(1 - s)^{\frac{\alpha_0}{4}} (1 + s)^{\frac{\alpha_0}{4}} b(s) \in L^1(-1, 1), \quad \text{for } \alpha_0 \in (0, 2]. \tag{0.15}$$

0.4 The weak formulation and Conservation Law.

To derive the weak formulation, a universal tool (so-called *Pre-postcollisional change of variables*) is frequently used, which is an involutive change of variables with unit Jacobian,

$$(v, v_*, \sigma) \to (v', v'_*, \hat{q}), \tag{0.16}$$

where $\hat{q}$ is the unit vector along the relative velocity $q := v - v_*$,

$$\hat{q} = \frac{v - v_*}{|v - v_*|}. \tag{0.17}$$

On the other hand, since $\sigma = (v' - v'_*)/|v' - v'_*|$, the change of variables (0.16) formally amounts to the change of (v, v_*) and (v', v'_*). Hence, under suitable integrability conditions on the measurable function F,

$$\begin{aligned}
&\int_{\mathbb{S}^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} B(|v - v_*|, \hat{q}\cdot\sigma) F(v, v_*, v', v'_*)\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\sigma \\
=&\int_{\mathbb{S}^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} B(|v - v_*|, \hat{q}\cdot\sigma) F(v, v_*, v', v'_*)\,\mathrm{d}v'\,\mathrm{d}v'_*\,\mathrm{d}\sigma \\
=&\int_{\mathbb{S}^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} B(|v - v_*|, \hat{q}\cdot\sigma) F(v', v'_*, v, v_*)\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\sigma,
\end{aligned} \tag{0.18}$$

where the fact $|v' - v'_*| = |v - v_*|, \sigma\cdot\hat{q} = \hat{q}\cdot\sigma$ is used to keep the arguments of collision kernel $B(v - v_*, \sigma) = B(|v - v_*|, \hat{q}\cdot\sigma)$ unchanged. Note that the change of variables $(v, v_*) \to (v', v'_*)$ works for a fixed ω but is illegal for any given σ.

With the help of this microreversiblity of velocity from (v, v) to (v', v'_*), which leaves the collision kernel B invariant, we can obtain the following weak form for the Boltzmann collision operator.

Proposition 0.1. *For any test function ϕ that is an arbitrarily continuous function of the velocity v,*

$$\begin{aligned}
\int_{\mathbb{R}^3} Q(f,f)\phi\,\mathrm{d}v &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v - v_*, \sigma)(f'f'_* - ff_*)\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v \\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v - v_*, \sigma) ff_*(\phi' - \phi)\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v \\
&= \frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v - v_*, \sigma) ff_*(\phi' + \phi'_* - \phi - \phi_*)\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v \\
&= \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v - v_*, \sigma)(f'f'_* - ff_*)(\phi + \phi_* - \phi' - \phi'_*)\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v.
\end{aligned} \tag{0.19}$$

Hence, there is an immediate consequence for a solution f to the Boltzmann equation that, whenever ϕ satisfies the functional equation,

$$\forall (v, v_*, \sigma) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2, \qquad \phi(v') + \phi(v'_*) = \phi(v) + \phi(v_*), \tag{0.20}$$

then, we at least formally have,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} f(t,v)\phi(v)\,\mathrm{d}v = 0 \tag{0.21}$$

and this kind of ϕ is usually called the *collision invariant*.

Since the mass, momentum and energy are conserved during the classical elastic collisions, it is natural to find that the functions $1, v_i, 1 \le i \le 3$, and $|v|^2$ and any linear combination of them are the collision invariants, which can be actually shown as the only collision invariants. Together with the weak form, this leads to the formal conservation law of the Boltzmann equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} f(t,v)\begin{pmatrix} 0 \\ v_i \\ |v|^2 \end{pmatrix}\mathrm{d}v = \int_{\mathbb{R}^3} Q(f,f)(t,v)\begin{pmatrix} 0 \\ v_i \\ |v|^2 \end{pmatrix}\mathrm{d}v = 0, \qquad 1 \le i \le 3. \tag{0.22}$$

In particular, at a given time t, one can define the local density ρ, the local macroscopic velocity u, and the local temperature T, by

$$\rho = \int_{\mathbb{R}^3} f(t,v)\,\mathrm{d}v, \quad \rho u = \int_{\mathbb{R}^3} f(t,v)v\,\mathrm{d}v, \quad \rho|u|^2 + d\rho T = \int_{\mathbb{R}^3} f(t,v)|v|^2\,\mathrm{d}v, \tag{0.23}$$

then the equilibrium is the *Maxwellian* distribution,

$$\mathcal{M}(v) = \mathcal{M}^f(v) = \frac{1}{(2\pi T)^{3/2}}\,\mathrm{e}^{-\frac{|v-u|^2}{2T}}. \tag{0.24}$$

0.5 Boltzmann's H–Theorem and Equilibrium

If not caring about the integrability issues, we select the test function $\phi = \log f$ into the weak form (0.19), and consider the properties of the logarithm function, to find that

$$\begin{aligned} -\int_{\mathbb{R}^3} Q(f,f)\ln f\,\mathrm{d}v =& D(f) \\ =& \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-v_*,\sigma)(f'f'_* - ff_*)\ln\frac{f'f'_*}{ff_*} \geq 0 \end{aligned} \tag{0.25}$$

due to the fact that the function $(X,Y) \longmapsto (X-Y)(\ln X - \ln Y)$ is always non-negative. Thus, if we introduce Boltzmann's H–functional,

$$H(f) = \int_{\mathbb{R}^3} f\ln f\,\mathrm{d}v \tag{0.26}$$

then the $H(f)$ will evolve in time because of the collisional effect that

$$\frac{\mathrm{d}}{\mathrm{d}t}H(f(t,\cdot)) = -D(f(t,\cdot)) \leq 0, \tag{0.27}$$

which is the well-known Boltzmann's H–Theorem: the H–functional, or entropy, is non-increasing with time evolution.

And the equality holds if and only if $\ln f$ is a collision invariant, i.e., $f = \exp(a + bv + c|v|^2)$ with a, b, c being all constants.

0.6 Fourier Transform of the Probability Measures

To give a accurate definition to the space of probability measures, we first define, if $\Omega = \mathbb{R}^3$ (a locally compact set which is not compact),

$$C_0(\mathbb{R}^3) := \left\{\phi(v) \in C(\mathbb{R}^3);\ \lim_{|v|\to\infty}\phi(v) = 0\right\} \tag{0.28}$$

with

$$C_0(\mathbb{R}^3) = \overline{\mathcal{D}(\mathbb{R}^3)}^{\|\cdot\|_\infty}. \tag{0.29}$$

Then, the space of Radon measures defined as

$$M(\mathbb{R}^3) := \left\{\mu : C_0 \longmapsto \mathbb{R};\ \mu \text{ is linear s.t. } \exists C > 0,\ |\mu(\phi)| \leq C\|\phi\|_\infty,\ \forall\phi \in \mathcal{D}(\mathbb{R}^3)\right\} \tag{0.30}$$

associated with the measure norm

$$\|\mu\|_{M(\mathbb{R}^3)} := \sup_{\phi\in\mathcal{D}(\mathbb{R}^3),\|\phi\|_\infty\leq 1} |\mu(\phi)|\,. \tag{0.31}$$

is a Banach space.

Remark 0.2. *(i) Another way to define the sapce M could be to replace $C_0(\mathbb{R}^3)$ by $C(\mathbb{R}^3_c)$, where $\mathbb{R}^3_c$ is denoted as the compactification of $\mathbb{R}^3$ by means of a single point ∞, implying that the limit value $\phi(\infty)$ exists for any $\phi \in C(\mathbb{R}^3_c)$. This is a technical issue that we need to have convenient compactness properties for some subsets of $M(\mathbb{R}^3_c)$.*

(ii) The functionals $\mu \in M(\mathbb{R}^3)$ are called (Radon) measures, since there is a one-to-one correspondence between elements of $M(\mathbb{R}^3)$ and a class of (Borel) measures $\tilde{\mu}$ on $\mathbb{R}^3$ with finite total mass $\tilde{\mu}(\mathbb{R}^3) < 0$, such that

$$\mu(\phi) = \int_{\mathbb{R}^3} \phi \,\mathrm{d}\tilde{\mu}, \qquad \forall \phi \in C_0(\mathbb{R}^3). \tag{0.32}$$

And, as usual, we do not distinguish between μ and $\tilde{\mu}$. Hence, instead of $\mu(\phi)$, we use the standard duality notation

$$\langle \mu, \phi \rangle := \mu(\phi) = \int_{\mathbb{R}^3} \phi \,\mathrm{d}\mu, \quad \mu \in M(\mathbb{R}^3),\ \phi \in C_0(\mathbb{R}^3). \tag{0.33}$$

If $\mu \in M(\mathbb{R}^3), \mu(\phi) > 0$ for all $0 \leq \phi \in \mathcal{D}(\mathbb{R}^3)$, we say that μ belongs to a non-negative bounded Radon measure space $M_+(\mathbb{R}^3)$.

Finally, the space of *probability measure* is defined as follows:

$$P(\mathbb{R}^3) := \left\{ \mu \in M_+(\mathbb{R}^3) \text{ with } \|\mu\|_{M(\mathbb{R}^3)} = 1 \right\}. \tag{0.34}$$

Proposition 0.3. *For $\mu \in P(\mathbb{R}^3)$, we define the Fourier transform by*

$$\mathcal{F}(\mu)(\xi) = \hat{\mu}(\xi) := \int_{\mathbb{R}^3} \mathrm{e}^{-\,\mathrm{i} v\cdot\xi} \,\mathrm{d}\mu(v). \tag{0.35}$$

Then, $\varphi(\xi) = \hat{\mu}(\xi)$ is called the characteristic function and

$$\mathcal{F} : P(\mathbb{R}^3) \longmapsto C_b(\mathbb{R}^3), \tag{0.36}$$

where $C_b(\mathbb{R}^3)$ is the space of bounded continuous functions.

Proof. Consider, for $\mu \in P(\mathbb{R}^3)$,

$$|\hat{\mu}(\xi)| \leq \left| \int_{\mathbb{R}^3} \mathrm{e}^{-\,\mathrm{i} v\cdot\xi} \,\mathrm{d}\mu(v) \right| \leq \int_{\mathbb{R}^3} \left|\mathrm{e}^{-\,\mathrm{i} v\cdot\xi}\right| \mathrm{d}\mu(v) = 1 < \infty, \tag{0.37}$$

where we observe $\left|\mathrm{e}^{-\,\mathrm{i} v\cdot\xi}\right| \leq 1$ and the continuity of $\hat{\mu}(\xi)$ follows from the dominated convergence theorem. □

Proposition 0.4. *Let $\mu \in P(\mathbb{R}^3)$, then μ is uniquely determined by $\hat{\mu}$.*

Proof. We need to show that $\mu_1 = \mu_2$, if $\hat{\mu}_1 = \hat{\mu}_2$. For any $\phi \in \mathcal{S}$, we have

$$\int_{\mathbb{R}^3} \hat{\phi}(v) \,\mathrm{d}\mu(v) = \int_{\mathbb{R}^3} \phi(\xi)\hat{\mu}(\xi) \,\mathrm{d}\xi. \tag{0.38}$$

Indeed, by the Fubini's theorem,

$$\int_{\mathbb{R}^3} \hat{\phi}(v) \,\mathrm{d}\mu(v) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \mathrm{e}^{-\,\mathrm{i}\xi\cdot v} \phi(\xi) \,\mathrm{d}\xi \,\mathrm{d}\mu(v) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \mathrm{e}^{-\,\mathrm{i}\xi\cdot v} \,\mathrm{d}\mu(v)\phi(\xi) \,\mathrm{d}\xi = \int_{\mathbb{R}^3} \hat{\mu}(\xi)\phi(\xi) \,\mathrm{d}\xi \tag{0.39}$$

Furthermore, if $\mu_1(v) = \mu_2(v)$, then by (0.38), we find,

$$\int_{\mathbb{R}^3} \hat{\phi}(v)\, \mathrm{d}\mu_1(v) = \int_{\mathbb{R}^3} \hat{\phi}(v)\, \mathrm{d}\mu_2(v) \tag{0.40}$$

for any $\phi \in \mathcal{S}$. Since the Fourier transform in invertible on $\mathcal{S}$, we can also write it as,

$$\int_{\mathbb{R}^3} \hat{\phi}(v)\, \mathrm{d}\mu_1(v) = \int_{\mathbb{R}^3} \hat{\phi}(v)\, \mathrm{d}\mu_2(v), \qquad \forall \phi \in \mathcal{S}. \tag{0.41}$$

By choosing the mollifier function $\phi_\epsilon \in \mathcal{S}$ such that

$$\mathbf{1}_{[a,b]} \leq \phi_\epsilon \leq \mathbf{1}_{[a-\epsilon, b-\epsilon]}, \tag{0.42}$$

it follows the dominated convergence theorem

$$\mu_1([a,b]) = \lim_{\epsilon \to 0} \int_{\mathbb{R}^3} \phi_\epsilon\, \mathrm{d}\mu_1(v) = \lim_{\epsilon \to 0} \int_{\mathbb{R}^3} \phi_\epsilon\, \mathrm{d}\mu_2(v) = \mu_2([a,b]) \tag{0.43}$$

that $\mu_1 = \mu_2$. □

Let μ_j and μ denote probability measure on $\mathbb{R}^3$, we say that μ_j convergent weakly to μ if

$$\lim_{j \to \infty} \int_{\mathbb{R}^3} \phi\, \mathrm{d}\mu_j = \int_{\mathbb{R}^3} \phi\, \mathrm{d}\mu \tag{0.44}$$

for all $\phi \in C_0(\mathbb{R}^3)$, which is the weak convergence of measures.

Proposition 0.5. *If μ_j and μ belong to $P(\mathbb{R}^3)$ and for each $\xi \in \mathbb{R}^3$,*

$$\lim_{j \to \infty} \hat{\mu}_j(\xi) = \hat{\mu}(\xi), \tag{0.45}$$

then,

$$\lim_{j \to \infty} \phi(v)\, \mathrm{d}\mu_j(v) = \int_{\mathbb{R}^3} \phi(v)\, \mathrm{d}\mu(v), \tag{0.46}$$

for all $\phi \in \mathcal{S}(\mathbb{R}^3)$.

Proof. Since $\mu_j \in P(\mathbb{R}^3)$ is a probability measure with $\mu_j(\mathbb{R}^3) = 1$, we have,

$$\sup_j |\hat{\mu}_j(\xi)| = \sup_j \mu_j(\mathbb{R}^3) = 1 \leq \infty. \tag{0.47}$$

Then, for $\phi \in \mathcal{S}$, $|\phi(\xi)\hat{\mu}_j(\xi)| \leq |\phi(\xi)|$, hence, it follows the dominated convergence theorem that,

$$\lim_{j \to \infty} \int_{\mathbb{R}^3} \phi(\xi)\hat{\mu}_j(\xi)\, \mathrm{d}\xi = \int_{\mathbb{R}^3} \phi(\xi)\hat{\mu}(\xi)\, \mathrm{d}\xi, \tag{0.48}$$

which is equivalent to, by Fubini's theorem,

$$\lim_{j \to \infty} \int_{\mathbb{R}^3} \hat{\phi}(v)\, \mathrm{d}\mu_j(v) = \int_{\mathbb{R}^3} \hat{\phi}(v)\, \mathrm{d}\mu(v). \tag{0.49}$$

Finally, the desired weak convergence of μ_j to μ follows from the fact that the set of all $\hat{\phi}$ such that $\phi \in \mathcal{S}$ is all of $\mathcal{S}$. □

0.7 Fourier Transform of the Collision Operator (Bobylev Identity)

The Fourier transformation has been widely used in the analysis of various kind of partial differential equations. However, it used to be very painful to find an elegant representation of the Boltzmann equation in the Fourier space, even though the Boltzmann operator possesses a nice weak formulation. Thanks to A. V. Bobylev, this problem turned out not as intricate as one may imagine, at least for the Maxwellian molecules. Since then, the so-called "Bobylev Identity" has become an extremely powerful technique in the study of the Boltzmann equation, especially in the case of spatially homogeneous theory.

Proposition 0.6. *Consider the Boltzmann collision operator $Q(g,f)$ with its collision kernel B being the Maxwellian molecule b, i.e., B does not depend on $|v-v_*|$,*

$$Q(f,f)(v) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b\left(\frac{v-v_*}{|v-v_*|}\cdot\sigma\right)\left[f(v')f(v'_*) - f(v)f(v_*)\right]\mathrm{d}\sigma\,\mathrm{d}v_*. \tag{0.50}$$

Then, the following formulas hold,

$$\begin{aligned}\mathcal{F}\left[Q^+(g,f)\right](\xi) &= \int_{\mathbb{S}^2} b\left(\frac{\xi}{|\xi|\cdot\sigma}\right)\hat{g}(\xi^-)\hat{f}(\xi^+)\,\mathrm{d}\sigma,\\ \mathcal{F}\left[Q^-(g,f)\right](\xi) &= \int_{\mathbb{S}^2} b\left(\frac{\xi}{|\xi|\cdot\sigma}\right)\hat{g}(0)\hat{f}(\xi)\,\mathrm{d}\sigma,\end{aligned} \tag{0.51}$$

where,

$$\xi^+ = \frac{\xi}{2} + \frac{|\xi|}{2}\sigma, \qquad \xi^- = \frac{\xi}{2} - \frac{|\xi|}{2}\sigma. \tag{0.52}$$

Proof. By performing the weak formulation, for any test function ϕ, we have,

$$\int_{\mathbb{R}^3} Q^+(g,f)(v)\phi(v)\mathrm{d}v = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b\left(\frac{v-v_*}{|v-v_*|}\cdot\sigma\right) g(v_*)f(v)\phi(v')\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v. \tag{0.53}$$

Selecting $\phi(v) = \mathrm{e}^{-\mathrm{i}v\cdot\xi}$ in the identity above, we have

$$\begin{aligned}&\mathcal{F}\left[Q^+(g,f)\right](\xi)\\ &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b\left(\frac{v-v_*}{|v-v_*|}\cdot\sigma\right) g(v_*)f(v)\,\mathrm{e}^{-\mathrm{i}\left(\frac{v+v_*}{2}+\frac{|v-v_*|}{2}\sigma\right)\cdot\xi}\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v\\ &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b\left(\frac{v-v_*}{|v-v_*|}\cdot\sigma\right) g(v_*)f(v)\,\mathrm{e}^{-\mathrm{i}\frac{v+v_*}{2}\cdot\xi}\,\mathrm{e}^{-\mathrm{i}\frac{|v-v_*|}{2}\sigma\cdot\xi}\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v,\end{aligned} \tag{0.54}$$

according to the general change of variable,

$$\int_{\mathbb{S}^2} F(k\cdot\sigma, l\cdot\sigma)\,\mathrm{d}\sigma = \int_{\mathbb{S}^2} F(l\cdot\sigma, k\cdot\sigma)\,\mathrm{d}\sigma, \quad |l| = |k| = 1, \tag{0.55}$$

due to the existence of an isometry on $\mathbb{S}^2$ exchanging l and k, we have, by exchanging the rule of $\frac{\xi}{|\xi|}$ and $\frac{v-v_*}{|v-v_*|}$,

$$\begin{aligned}&\int_{\mathbb{S}^2} g(v_*)f(v)b\left(\frac{v-v_*}{|v-v_*|}\cdot\sigma\right)\mathrm{e}^{-\mathrm{i}\frac{|v-v_*|}{2}\sigma\cdot\xi}\,\mathrm{d}\sigma\\ &= \int_{\mathbb{S}^2} g(v_*)f(v)b\left(\frac{\xi}{|\xi|}\cdot\sigma\right)\mathrm{e}^{-\mathrm{i}\frac{|\xi|}{2}\sigma\cdot(v-v_*)}\,\mathrm{d}\sigma\end{aligned} \tag{0.56}$$

Thus,

$$
\begin{aligned}
&\mathcal{F}\left[Q^+(g,f)\right](\xi)\\
&=\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} g(v_*)f(v)b\left(\frac{v-v_*}{|v-v_*|}\cdot\sigma\right)\mathrm{e}^{-\,\mathrm{i}\frac{v+v_*}{2}\cdot\xi}\,\mathrm{e}^{-\,\mathrm{i}\frac{|v-v_*|}{2}\sigma\cdot\xi}\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v\\
&=\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} g(v_*)f(v)b\left(\frac{\xi}{|\xi|}\cdot\sigma\right)\mathrm{e}^{-\,\mathrm{i}\frac{v+v_*}{2}\cdot\xi}\,\mathrm{e}^{-\,\mathrm{i}\frac{|\xi|}{2}\sigma\cdot(v-v_*)}\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v\\
&=\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} g(v_*)f(v)b\left(\frac{\xi}{|\xi|}\cdot\sigma\right)\mathrm{e}^{-\,\mathrm{i}v\cdot\left(\frac{\xi}{2}+\frac{|\xi|}{2}\sigma\right)}\,\mathrm{e}^{-\,\mathrm{i}v_*\cdot\left(\frac{\xi}{2}-\frac{|\xi|}{2}\sigma\right)}\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v\\
&=\int_{\mathbb{S}^2} b\left(\frac{\xi}{|\xi|}\cdot\sigma\right)\hat{f}(\xi^+)\hat{g}(\xi^-)\,\mathrm{d}\sigma,
\end{aligned}
\tag{0.57}
$$

where, unlike the elastic case, the ξ^+ and ξ^- are defined as

$$\xi^+=\frac{\xi}{2}+\frac{|\xi|}{2}\sigma,\qquad \xi^-=\frac{\xi}{2}-\frac{|\xi|}{2}\sigma. \tag{0.58}$$

And the formula for $\mathcal{F}\left[Q^-(g,f)\right](\xi)$ is then easily obtained by the same kind of but more simpler computations. $\square$

For a given probability measure $F:\mathbb{R}^3\longmapsto\mathbb{R}$ or its density function f, we define the corresponding characteristic function $\varphi(\xi)$ by the Fourier transform:

$$\varphi(\xi)=\hat{f}(\xi):=\int_{\mathbb{R}^3}\mathrm{e}^{-\,\mathrm{i}v\cdot\xi}f(v)\,\mathrm{d}v=\int_{\mathbb{R}^3}\mathrm{e}^{-\,\mathrm{i}v\cdot\xi}\,\mathrm{d}F(v), \tag{0.59}$$

and its inversion formula by normalization writes

$$f(v)=\int_{\mathbb{R}^3}\mathrm{e}^{\mathrm{i}v\cdot\xi}\hat{f}(\xi)\,\mathrm{d}\xi=\int_{\mathbb{R}^3}\mathrm{e}^{\mathrm{i}v\cdot\xi}\varphi(\xi)\,\mathrm{d}\xi. \tag{0.60}$$

As a consequence, the spatially homogeneous Boltzmann equation can be converted into the following equation for the new unknown function $\varphi=\varphi(t,\xi)$:

$$\partial_t\varphi(t,\xi)=\int_{\mathbb{S}^2} b\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\left[\varphi(t,\xi^+)\varphi(t,\xi^-)-\varphi(t,0)\varphi(t,\xi)\right]\,\mathrm{d}\sigma, \tag{0.61}$$

where the ξ^+ and ξ^- have the same definition as in (0.58), which also satisfy the following relations:

$$\xi^++\xi^-=\xi\quad\text{and}\quad|\xi^+|^2+|\xi^-|^2=|\xi|^2, \tag{0.62}$$

as well as

$$|\xi^+|^2=|\xi|^2\frac{1+\frac{\xi}{|\xi|}\cdot\sigma}{2}\quad\text{and}\quad|\xi^-|^2=|\xi|^2\frac{1-\frac{\xi}{|\xi|}\cdot\sigma}{2}. \tag{0.63}$$