MA1300_TB_Teachingnotes_Kunlun

Friday, November 27, 2020 1:15 PM

Chapter0

Tuesday, September 8, 2020 8:57 PM

MA1300 Tutorial Class TB1 session.

Tutor: QI Kunlun

E-mail: kunlun.qi@my.cityu.edu.hk

Office: Room 1392, FYW Building

Chapter 0, Chapter 1 $\Leftrightarrow$ self-practice #1.

"function" "limit"

1. (P21, #45) Find the domain and $\boxed{\text{sketch}}$ the graph of the function $(x, G(x))$

"set"

$$G(x) = \frac{3x + |x|}{x}$$ ← piecewise.

Domain: $\begin{cases} x = 0, & G(x) \text{ "DNE"} \\ x \neq 0, & G(x) \text{ well-defined.} \end{cases}$

(1) $(-\infty, 0) \cup (0, +\infty)$

(2) $\{x \in \mathbb{R} \mid x \neq 0\}$ "set" $\mathbb{R} \setminus \{0\}$

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

$$G(x) = \begin{cases} \frac{3x+x}{x} = 4, & x > 0 \\ 2x - x \ \ldots & x < 0 \end{cases}$$

$$x \quad G(x) = \begin{cases} \frac{3x+x}{x} = 4, & x>0 \\ \frac{3x-x}{x} = 2, & x<0 \end{cases}$$

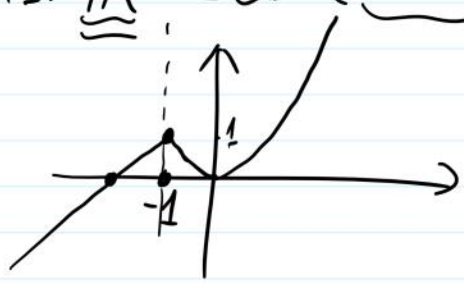
2. (P21, #49) Find the domain and sketch the graph of the function

$$f(x) = \begin{cases} x+2 & \text{if } x \le -1 \\ x^2 & \text{if } x > -1. \end{cases}$$

$(x, G(x))$

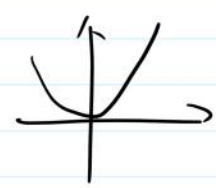
Pf: $\{x \mid x$ is all real number$\}$

(1) $\mathbb{R} \Leftrightarrow (-\infty, +\infty)$



$x = -1,\ f(-1) = 1.$

$x = -2,\ f(x) = 0$



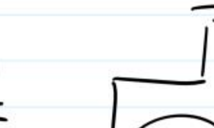
3. (P22, #67) In a certain country, income tax is assessed as follows. There is no tax on income up to \$10,000. Any income over \$10,000 is taxed at a rate of 10%, up to an income of \$20,000. Any income over \$20,000 is taxed at 15%.

(a) Sketch the graph of the tax rate R as a function of the income I.

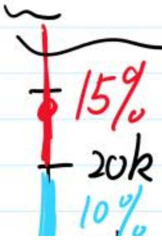
(b) How much tax is assessed on an income of \$14,000? On \$26,000?

(c) Sketch the graph of the total assessed tax T as a function of the income I.

(a)



$$R = \frac{\text{Tax}}{I} = \begin{cases} \frac{0}{I} = 0, & I \le 10k \\ \frac{(I-10{,}000)\times 10\%}{I}, & 10k < I \le 20k \\ \frac{(20{,}000-10{,}000)\times 10\% + (I-20{,}000)\times 15\%}{I}, & I > 20k. \end{cases}$$



$= \{ 0, \text{ if } I \le 10k$

20k

10%

10k

$$= \begin{cases} 0, & \text{if } I \le 10k \\ 0.1 - \frac{1000}{I}, & \text{if } 10k < I \le 20k. \\ 0.15 - \frac{2000}{I}, & \text{if } I > 20k. \end{cases}$$

(b) "Tax" $= R \cdot I =$

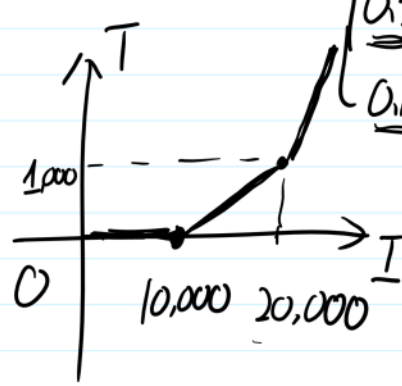
$I_1 = 14{,}000$, $R_1 = 0.1 - \frac{1000}{14{,}000}$, "Tax" $= R_1 \cdot I_1$

$I_1 = 14{,}000$, $R_1 = 0.1 - \frac{1000}{14{,}000}$, "Tax" $= R_1 \cdot I_1$
$= 400.$

$I_2 = 26{,}000$, $R_2 = 0.15 - \frac{2000}{26{,}000}$, "Tax" $= R_2 \cdot I_2$
$= 1900$

(c) $T = R \cdot I = R(I) \cdot I$

$$\overset{(a)}{=} \begin{cases} 0, & \text{if } I \le 10k. \\ 0.1 I - 1000, & \text{if } 10k < I \le 20k. \\ 0.15 I - 2000, & \text{if } I > 20k. \end{cases}$$



4. (P33, #13) The relationship between the Fahrenheit (F) and Celsius (C) temperature scales is given by the linear function $F = \frac{9}{5}C + 32$.

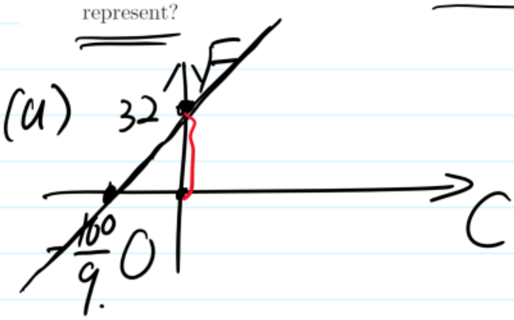
(a) Sketch a graph of this function.

"absolute zero"

(b) What is the slope of the graph and what does it represent? What is the F-intercept and what does it represent?

(b) What is the slope of the graph and what does it represent? What is the F-intercept and what does it represent?

(a)



$C=0$, $F(0)=32$.

$C=-\frac{160}{9}$, $F(C)=0$

(b) slope: $\frac{9}{5}$. the rate of change of "Dependent variable" with respect to "independent variable".

F

C

F-intercept: 32.

$C=0$ is equivalent to $F=32$

$C=0$ is equivalent to $F=32$

"$0°C$" "$32°F$"

"Freeze point"

5. (P34, #15) Biologists have noticed that the chirping rate of crickets of a certain species is related to temperature, and the relationship appears to be very nearly linear. A cricket produces 112 chirps per minutes at $20°C$ and 180 chirps per minute at $29°C$.

(a) Find a linear equation that models the temperature T as a function of the number of chirps per minute N.

(b) What is the slope of the graph? What does it represent?

(c) If the crickets are chirping at 150 chirps per minute, estimate the temperature.

(a). $T = aN + b$

$$\begin{cases} 20 = 112a + b \\ 29 = 180a + b \end{cases} \Rightarrow \begin{cases} a = \frac{9}{68} \\ b = \frac{88}{17} \end{cases}$$

$$\Rightarrow T = \frac{9}{68}N + \frac{88}{17}.$$

$129 = 180a + b \qquad b = \frac{88}{17}$

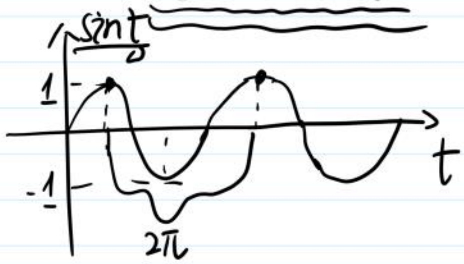
(b) slope: $\frac{9}{68}$. the rate of change of T with respect to N.

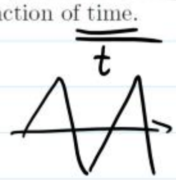
(C) $N = 150$.

$$T = \frac{9}{68} \times 150 + \frac{88}{17} \approx 25^{\circ}C$$

$\boxed{6.}$ (P43, #26) A variable star is one whose brightness alternately increases and decreases. For the most visible variable star, Delta Cephei, the time between $\boxed{\text{periods}}$ of maximum brightness is $\boxed{5.4 \text{ days}}$, the average brightness (or magnitude) of the star is 4.0, and its brightness varies by ± 0.35 magnitude. Find a function that models the brightness of Delta Cephei as a function of time.

amplitude.



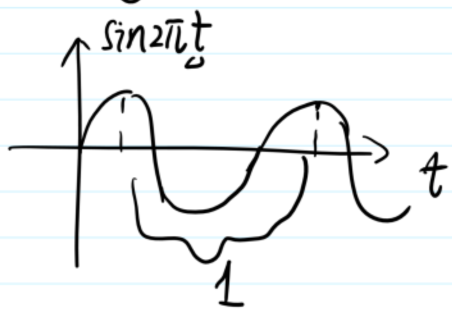


$$B = 4.0 + 0.35 \sin\left(\frac{2\pi t}{5.4}\right)$$

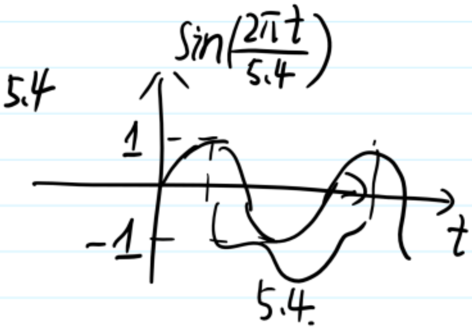
$-1 \quad 2\pi$

$$B = 4.0 + 0.35 \sin\left(\frac{2\pi t}{5.4}\right).$$

period $= 2\pi$. → 1



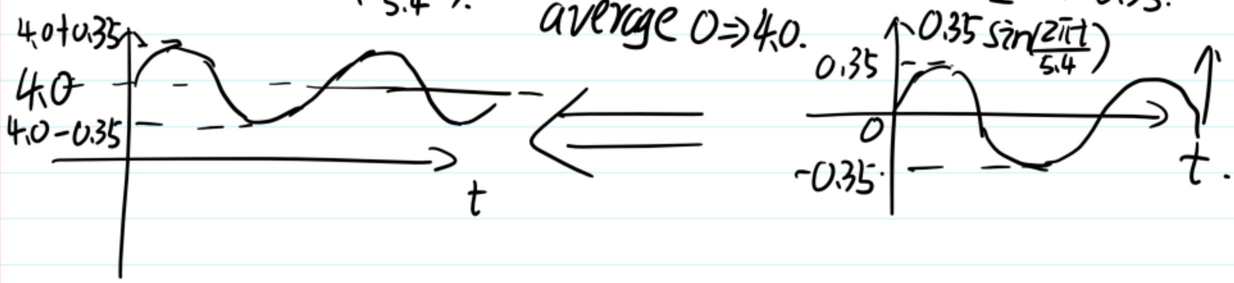
period $1 \Rightarrow 5.4$



amliplitude

$4.0 + 0.35\sin(\frac{2\pi t}{5.4})$. average $0 \Rightarrow 4.0$.

amlipltitude $1 \Rightarrow 0.35$.



7. (P44, #53) A stone is dropped into a lake, creating a circular ripple that travels outward at a speed of 60 cm/s.

(a) Express the radius r of this circle as a function of the time t (in seconds).

(b) If A is the area of this circle as a function of the radius, find $A \circ r$ and interpret it.

(a) $r = \text{velocity} \times \text{time}$.

$= 60t$

(b) $A \circ r = A(r(t)) = \pi r^2 = \pi(60t)^2 = 3600\pi t^2$.

Area of circular ripple at time t.

8. (P44, #57) The **Heaviside function** is defined by

$$H(t) = \begin{cases} 0 & \text{if } t < 0, \\ 1 & \text{if } t \geq 0. \end{cases}$$

It is used in the study of electric circuits to represent the sudden surge of electric current, or voltage, when a switch is instantaneously turned on.

It is used in the study of electric circuits to represent the sudden surge of electric current, or voltage, when a switch is instantaneously turned on.

(a) Sketch the graph of the Heaviside function.

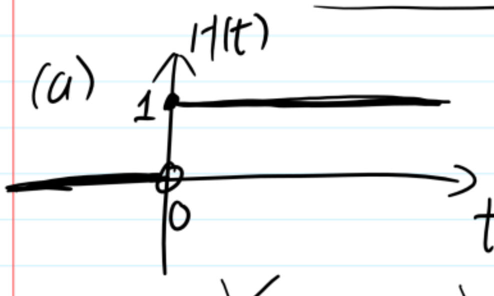
(b) Sketch the graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 0$ and 120 volts are applied instantaneously to the circuit. Write a formula for $V(t)$ in terms of $H(t)$.

(c) Sketch the graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 5$ seconds and 240 volts are applied instantaneously to the circuit. Write a formula for $V(t)$ in terms of $H(t)$. (Note that starting at $t = 5$ corresponds to a translation.)

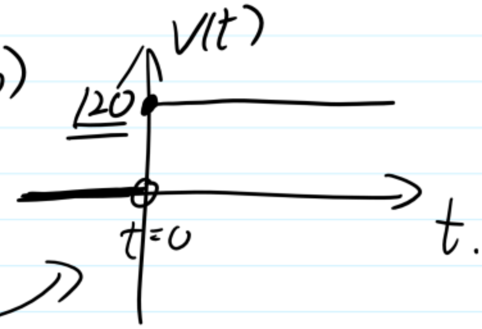
$H(t)$ $V(t)$

that starting at $t = 5$ corresponds to a translation.)

(a)

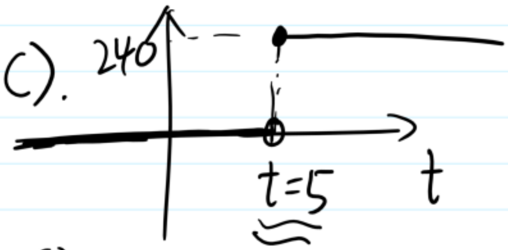


(b)



$$V(t) = 120H(t)$$

(C).



$$V(t) = \cancel{240H(t)}$$
$$= 240H(t-5)$$

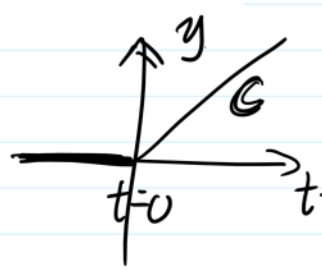
9. (P44, #58) The Heaviside function defined in the previous exercise can also be used to define the **ramp function** $y = ctH(t)$, which represents a gradual increase in voltage or current in a circuit. "$c>0$"

(a) Sketch the graph of the ramp function $y = ctH(t)$.

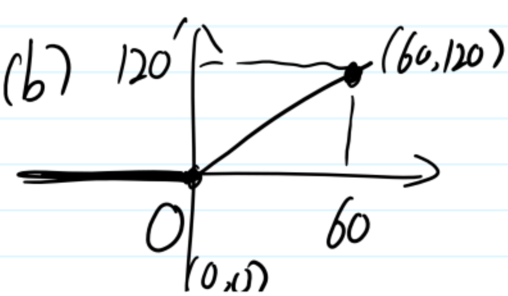
(b) Sketch the graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 0$ and the voltage is gradually increased to 120 volts over a 60-second time interval. Write a formula for $V(t)$ in terms of $H(t)$ for $t \leq 60$.

(c) Sketch the graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 7$ seconds and the voltage is gradually increased to 100 volts over a period of 25 seconds. Write a formula for $V(t)$ in terms of $H(t)$ for $t \leq 32$.

(a) $$y = ctH(t) = \begin{cases} 0, & t<0 \\ ct. & t \geq 0 \end{cases}$$



(b)

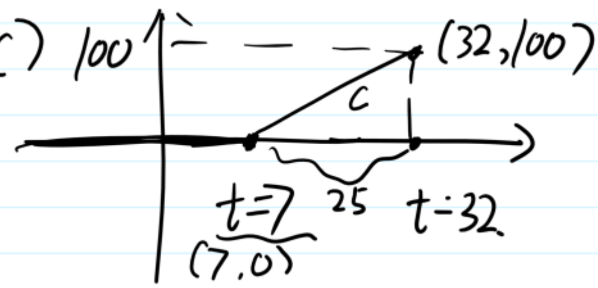


$$c = \frac{120-0}{60-0} = 2$$

$$V(t) = 2tH(t)$$

0 (0,0) 60 $V(t) = 2tH(t)$

(0,0)

(c) 

$$c = \frac{100-0}{32-7} = 4.$$

$$V(t) = \cancel{4t\,H(t)}$$

$$= 4(t-7)\,H(t-7).$$

10. (P44, #61)

(a) If $g(x) = 2x+1$ and $h(x) = 4x^2+4x+7$, find a function f such that $f \circ g = h$. (Think about what operations you would have to perform on the formula for g to end up with the formula for h.)

(b) If $f(x) = 3x+5$ and $h(x) = 3x^2+3x+2$, find a function g such that $f \circ g = h$.

(a) $f\circ g = f(g(x)) = h(x) = \boxed{4x^2}+4x+7$

$g(x) = \boxed{2x+1}$

$= (2x+1)^2 + 6$

$(2x+1)^2 = 4x^2+4x+1$

$= [g(x)]^2 + 6.$

let $y = g(x)$

$f(y) = y^2 + 6 \Rightarrow f(x) = x^2+6.$

(b) $f\circ g = f(g(x)) = 3g(x) + 5.$

$= h(x) = 3x^2+3x+2.$

$\Rightarrow 3g(x)+5 = 3x^2+3x+2$

$\Rightarrow g(x) = x^2+x-1$

$\Rightarrow \quad g(x) = x^2 + x - 1$

Chapter1

Friday, September 18, 2020 11:45 AM

Self Practice #2.

$\frac{\to 0}{\to 0}$ $\to 0$ $\to +\infty$ constant $\frac{\infty}{\infty}$ 1^{∞}

1. $\lim_{t \to -3} \frac{t^2-9}{(2t^2+7t+3)(t+3)}$ $t=-3$. $t^2-9=0$ $2t^2+7t+3=0$

$\frac{\infty}{\infty}$ 1^{∞}

$y = x+1$

$\lim_{t \to x_0} = x_0+1$

$= \lim_{t \to -3} \frac{(t-3)\cancel{(t+3)}}{(2t+1)\cancel{(t+3)}^2} \longrightarrow +\infty$

$= \lim_{t \to -3} \frac{t-3}{(2t+1)(t+3)} = \frac{6}{5}$ $\Rightarrow +\infty$

"L'hospital" theorem

$\frac{0}{0}, \frac{\infty}{\infty}$ "Taylor expansion"

3. $\lim_{t \to 0} \left(\frac{1}{t} - \frac{1}{t^2+t}\right)$ $(\infty)-(\infty)$ "bigger" "quicker"

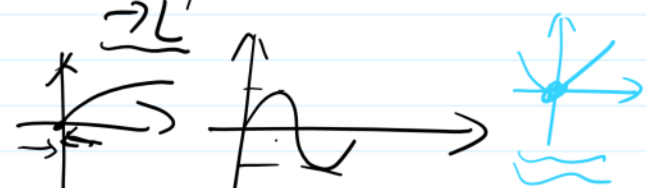
$= \lim_{t \to 0} \frac{t+1-1}{t(t+1)} = \lim_{t \to 0} \frac{1}{t+1} = \underline{\underline{1}}$

4. "Squeeze theorem"

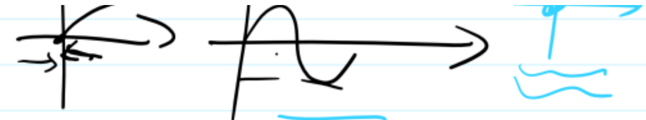
"$\sqrt{y}$" $\boxed{h(x)} \le f(x) \le \boxed{g(x)} \Rightarrow \lim f(x) = L$.

$\to L$ $\to L$

$\lim_{x \to 0} \sqrt{x^3+x^2} \sin\left(\frac{\pi}{x}\right) = 0$



$$\lim_{x\to 0} \sqrt{x^3+x^2}\,\sin\left(\frac{\pi}{x}\right) = 0$$



$$-\frac{1}{\sqrt{2}}|x| \le -\sqrt{x^3+x^2} \le \sqrt{x^3+x^2}\,\sin\frac{\pi}{x} \le \boxed{\sqrt{x^3+x^2}} \le \sqrt{\frac{3}{2}}\,|x|$$

$h(x)\to 0$. 'What happen' $x\to 0$ $\quad g(x)\to 0$

$-\frac{1}{2}\quad 0\quad \frac{1}{2}$

$$\lim_{x\to 0} \sqrt{x^3+x^2}\,\sin\frac{\pi}{x} = 0$$

$\sqrt{x^2(x+1)}$

$-\frac{1}{2}\quad 0\quad \frac{1}{2}$ $\quad x\to 0$ $\quad \sqrt{x^2(x+1)}$

$x\in(-\frac{1}{2},0)\cup(0,\frac{1}{2})$

$\boxed{\sqrt{x^2}=|x|}$ $\quad = |x|\sqrt{x+1}$ $\quad = |x|\sqrt{\frac{3}{2}}$ ✓

6. $\lim_{x\to 0^+} \sqrt{x}\,[1+\sin^2(\frac{2\pi}{x})] = 0$

$\ge 0 \quad \ge 0 \quad \ge 0$ $\qquad |-2| = 2$

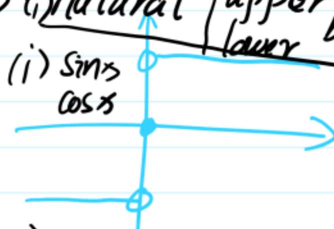
$\sin\frac{1}{x}$

$$\underbrace{0}_{\to 0} \le \sqrt{x}\,[1+\sin^2(\tfrac{2\pi}{x})] \le \underbrace{2\sqrt{x}}_{\to 0}$$

$\sqrt{x}\le$ $\quad \le 2$

Tip (i) natural upper / lower bound.

(i) $\sin x$ $\cos x$



10. $\lim_{x\to 0^+}|\text{sgn}(x)| = \lim_{x\to 0^-}|\text{sgn}(x)|$! $\quad \lim_{x\to 0} = 1$

$\parallel$ $\quad$ (a)

$$\text{sgn}(x) = \begin{cases} -1 & x<0 \\ 0, & x=0 \\ 1, & x>0 \end{cases}$$

$|\text{sgn}(0)|$ $\quad$ $|\text{sgn}(x)| = \begin{cases} 1, & x\ne 0 \\ 0, & x=0 \end{cases}$

$\parallel$

0

"continuous"

$\parallel$

"Limit exist" : r.h.l = l.h.l

(b). $\lim_{x\to 0^+} \text{sgn}\,x = \lim_{x\to 0^+} 1 = 1.$

$\lim_{x\to 0^-} \text{sgn}\,x = \lim_{x\to 0^-} -1 = -1.$

$$\lim_{x\to 0^-} \operatorname{sgn} x = \lim_{x\to 0^-} -1 = -1.$$

$$\lim_{x\to 0} \operatorname{sgn} x = \text{DNE}.$$

$$\lim_{x\to 0} |\operatorname{sgn} x| = 1 \neq |\operatorname{sgn}(0)| = 0.$$

$\Longrightarrow$ $|\operatorname{sgn} x|$ is NOT continuous at $x=0$.

7. $\lim_{x\to 3} (2x+|x-3|)$ "DNE" $\boxed{\text{continuous}} \Longrightarrow \lim_{x\to x_0} f(x) = f(x_0)$

$\boxed{8,9}$ lim Exist $\not\Rightarrow$ continuous

$\boxed{8,9}$ lim Exist $\not\Rightarrow$ continuous

Pf: when $x>3$, $2x+(x-3)=3x-3$, $\lim_{x\to 3^+} 3x-3 = 6$.

Check Limit Does Exist: when $x<3$, $2x-(x-3)=x+3$, $\lim_{x\to 3^-} x+3 = 6$.

$$\Longrightarrow \lim_{x\to 3} (2x+|x-3|) = 6.$$

11. $L = L_0\sqrt{1-v^2/c^2}$

$$\lim_{v\to c^-} L = \lim_{v\to c} L_0 \frac{1}{c}\sqrt{\underbrace{c^2-v^2}_{\geq 0}} = 0$$

$\boxed{\begin{matrix} v<c \\ v\to c \end{matrix}}$

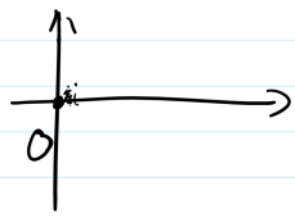
$L \to 0$, when the speed of object approach to speed of light

$\Sigma \to 0$, when the speed of object approach to speed of light.

Physically, No object faster light. $v < c$

Mathematically, $c^2 - v^2 \geq 0 \Rightarrow v \leq c$.

12. $f(x) = \begin{cases} x^2, & \text{if } x \text{ is rational.} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$



$\lim_{x \to 0} f(x) = 0.$

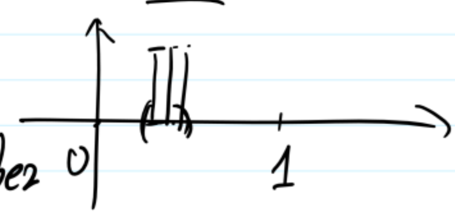
Dirichet function

$\Rightarrow \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$

$x \to 0$

$\boxed{0 \leq f(x) \leq x^2}$ for all real number

$x^2 \to 0$



$\Rightarrow$ Squeeze. theorem

$\lim_{x \to 0} f(x) = 0.$

Nowhere continuous

Thomae's function

P15 $f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$

Pf.

(1 if x is irrational

Pf: Prove that $\lim_{x\to 0} f(x)$ does NOT exist.

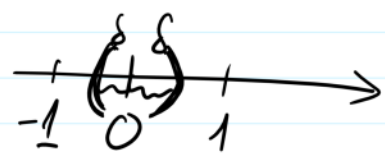
"δ-ε" language | $\boxed{\text{Exist}}$: $\forall \varepsilon>0, \exists \delta$, when $|x-x_0|<\delta$, then $|f(x)-L|<\varepsilon$.

constant

$\exists \varepsilon>0, \forall \delta>0$, when $|x-x_0|<\delta$, then $|f(x)-L|>\varepsilon$ "logical formally reverse"

Suppose $\lim_{x\to 0} f(x)=L$. $\exists$ "ε" $=\frac{1}{4}, \frac{1}{2}$. $|x-0|<\delta$

$\forall \delta>0$, when $|x-0|<\delta$,



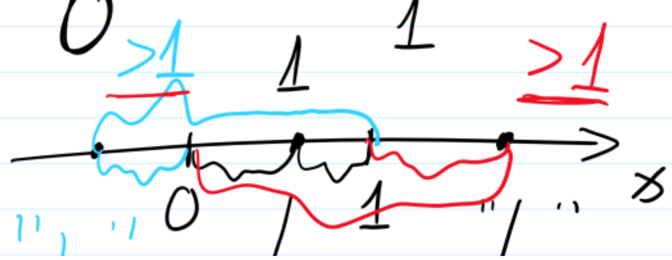
density of ration/irrational in the neighborhood $|x-0|<\delta$

$\downarrow x_\delta$ $\downarrow x'_\delta$

$|f(x_\delta)-L|+|f(x'_\delta)-L| > \varepsilon$

$$\underbrace{|f(x_\delta)-L|}_{0}+\underbrace{|f(x'_\delta)-L|}_{1} > \varepsilon=\frac{1}{4}, \text{or } \frac{1}{2}.$$

>1



"L" 0 L 1 "L" x

$\Longrightarrow \lim_{x\to 0} f(x)$ does NOT exist.

P13: $\displaystyle \lim_{x\to 2}\frac{\sqrt{6-x}-2}{\sqrt{3-x}-1}$ "$\frac{0}{0}$" $\qquad x^2-1 = (x-1)(x+1)$

$$= \lim_{x\to 2}\frac{(\sqrt{6-x}-2)(\sqrt{6-x}+2)(\sqrt{3-x}+1)}{(\sqrt{3-x}-1)(\sqrt{6-x}+2)(\sqrt{3-x}+1)}$$

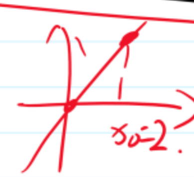
$$= \lim_{x\to 2}\frac{\cancel{(2-x)}(\sqrt{3-x}+1)}{\cancel{(2-x)}(\sqrt{6-x}+2)}$$

$$= \frac{1}{2}$$

14. Find δ such that if $|x-2|\le\delta$, then $|4x-8|<\varepsilon$, where $\varepsilon=0.1$.

$x_0=2$ $\qquad 4x_0=4\times 2=8$

$f(x)$ is continuous at $x=x_0$:



$(\forall\varepsilon>0,\ \exists\delta>0$, when $|x-x_0|\le\delta$, then

$(\forall\varepsilon>0,\ \exists\delta>0$, when $|x-x_0|\le\delta$, then $|f(x)-f(x_0)|<\varepsilon.)$

$|f(x) - f(x_0)| < \varepsilon.$

$\delta = \frac{1}{80}, \frac{1}{100}, \frac{1}{1000}$

$|x-2| < \delta.$ $|4x-8| = 4|x-2| = 4\delta = \frac{4}{80} < 0.1$

17. $\lim\limits_{x \to -3} \frac{1}{(x+3)^4} = \infty.$ " $\frac{1}{0} = \infty$ "

" L "

Pf: " $\varepsilon - \delta$ " language $\lim\limits_{x \to x_0} f(x) = \infty.$

$\forall M > 0$

For any $M > 0$,

$\exists \delta > 0$, when $|x - x_0| < \delta$,

then, we have $f(x) > M$

For any big enough $M > 0$,

$\delta = \left(\left(\frac{1}{2M}\right)^{\frac{1}{4}}\right)$, when $|x-(-3)| < \delta$, we have

$\frac{1}{(x+3)^4} \geq M$

$|x+3| < \delta.$

$\Downarrow$

$|x+3|^4 < \delta^4$

$\Downarrow$

$\frac{1}{|x+3|^4} > \frac{1}{\delta^4} > M.$

Tip: Critical point.

" $\frac{1}{\left(\sqrt[4]{\frac{1}{2M}}\right)^4} \geq 2M > M$

$|x+3|^4 < (\delta^4) < |M|.$

$\Rightarrow \delta < \sqrt[4]{\frac{1}{M}} \quad \left(\frac{1}{M}\right)^{\frac{1}{4}}$

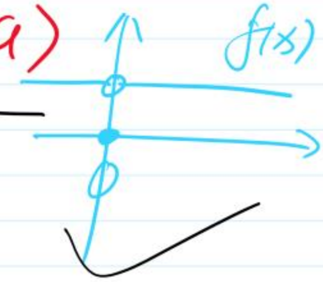
$\delta = \left(\frac{1}{2M}\right)^{\frac{1}{4}}$

Self-Practice #3.

1. (1) $f(x) = x^2 + \sqrt{7-x}$, $a=4$.

Continuous at given point $a=4$

$\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = f(a)$



limit exist at $a=4$.

$\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = \lim_{x\to a} f(x)$

$\lim_{x\to 4^+} x^2 + \sqrt{7-x} = 16 + \sqrt{3}$

$f(4) = 16 + \sqrt{3}$

$\lim_{x\to 4^-} x^2 + \sqrt{7-x} = 16 + \sqrt{3}$

$x<4,\ x\to 4.$

3. $f(x) = \begin{cases} 1-x^2, & \text{if } x<1. \\ \frac{1}{x}, & \text{if } x\geq 1. \end{cases}$ $a=1.$

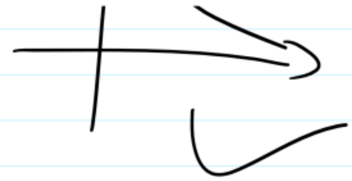
① check ...

① check $x>1$, $x<1$.

① check $x>1$, $x<1$.

$\Downarrow$ $\frac{1}{x}$ $\qquad$ $\Downarrow$ $1-x^2$

continuous function.



Conclude that

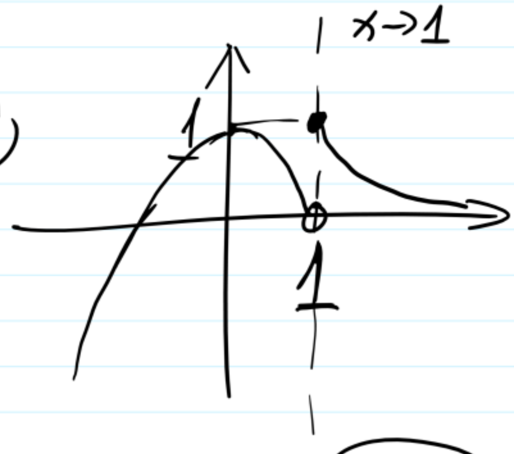
$f(x)$ is NOT continuous at $x=1$.

② check $a=1$.

$$\lim_{x\to 1^+} f(x) = \lim_{\substack{x>1\\ x\to 1}} \frac{1}{x} = 1 = f(1)=1$$

$$\lim_{x\to 1^-} f(x) = \lim_{\substack{x<1\\ x\to 1}} 1-x^2 = 0.$$

③



5 (1) $\lim_{x\to\pi} \sin(x+\sin x)$

$= \sin\left(\lim_{x\to\pi} (x+\sin x)\right)$

① check domain of $\sin y$ is $\mathbb{R}$.

$= \sin\left(\lim_{x\to\pi} (x + \sin x)\right)$ ... is π.

$= \sin(\pi) = 0.$

(2) $\lim_{x\to 2} (x^3 - 3x + 1)^{-3}$ ① check domain of $\frac{1}{y^3}$ is $\mathbb{R}\setminus\{0\}$

let $y = x^3 - 3x + 1.$ of $\frac{1}{y^3}$ is $\mathbb{R}\setminus\{0\}$

$\lim_{x\to 2} y(x) = 3.$

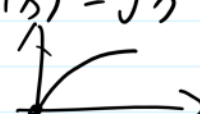
$\Rightarrow \lim_{x\to 2} (x^3 - 3x + 1)^{-3} = \frac{1}{27}.$

6. f is continous on $\overset{\mathbb{R}}{(-\infty, +\infty)}$

$$f(x) = \begin{cases} x^2, & \text{if } x < 1 \\ \sqrt{x}, & \text{if } x \geq 1. \end{cases} \qquad x = 1$$

① step 1: if $x < 1$, $f(x) = x^2$, continous.

if $x > 1$. $f(x) = \sqrt{x}$, continous.



② step 2: focus on $x = 1$.

② Step 2: focus on $x=1$.

$$\lim_{x\to 1^-} f(x) \overset{?}{=} \lim_{x\to 1^+} f(x) \overset{?}{=} f(1)$$

$$\lim_{\substack{x\to 1\\ x<1}} x^2 = 1, \quad \lim_{\substack{x\to 1\\ x>1}} \sqrt{x} = 1, \quad f(1) = 1$$

$1 = 1$

conclude $f(x)$ continous at $x=1$.

③ combined ① and ②, conclude that $f(x)$ continous on $(-\infty, +\infty)$

7. $$F(r) = \begin{cases} \dfrac{GMr}{R^3}, & \text{if } r < R \\ \dfrac{GM}{r^2}, & \text{if } r \geq R. \end{cases}$$

mass of earth (M)

radius of earth (R)

Cr ✓

$\dfrac{C'}{r^2}$ ✓

$\{\mathbb{R} \mid r \neq 0\}$

Is F a continous of r.

① if $r<R$, ~~if $r>R$~~ > continuous

② "R" $\lim_{r\to R^-} F(r) \overset{?}{=} \lim_{r\to R^+} F(r) \overset{?}{=} F(R)$

$4c+4=8-2c$

$\Rightarrow \; c=\frac{2}{3}.$

$\underset{x\to 2^-}{} \qquad \underset{x\to 2^+}{} \qquad$

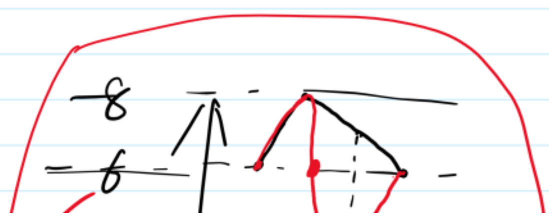
$\qquad \| \qquad\qquad \| \qquad\qquad \|$

$4c+4 = 8-2c = 8-2c$

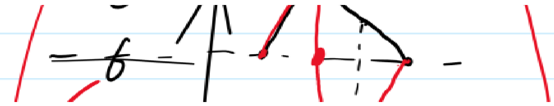
10 (1) Suppose f is continuous on $[1,5]$.

(2) Only solutions to equation $f(x)=6$ are $x=1$ and $x=4$.

(3) If $f(2)=8$.



③ If $f(2) = 8$.



~~Condition (2)~~. Contradiction.

$\Longrightarrow f(3) > 6$.

---

self-practice #4.

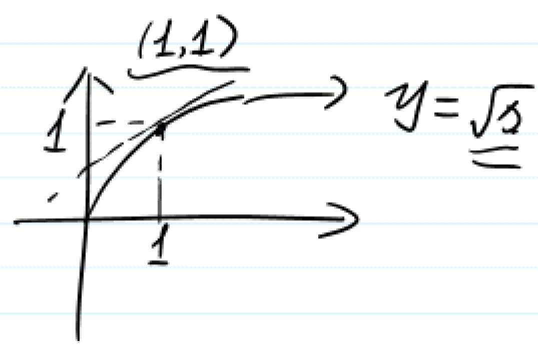
P1: $f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & \text{if } x<2 \\ ax^2-bx+3 & \text{if } 2 \le x < 3 \end{cases}$ continuous "$\mathbb{R}$"

Chapter2

Wednesday, October 28, 2020 1:01 PM

Self-practice #5.

1. tangent line $\Rightarrow$ point, slope



$y-1 = k(x-1)$

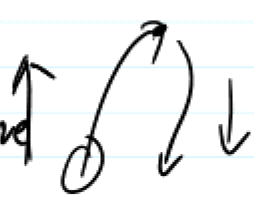
k $\Leftrightarrow$ Derivative $y'(x) = \frac{dy}{dx}\Big|_{x=1} = \frac{1}{2\sqrt{x}}\Big|_{x=1} = \frac{1}{2}$.

$\Rightarrow$ equation: $y-1 = \frac{1}{2}(x-1)$.

2. $H = 10t - 1.86t^2$

H $\Rightarrow$ height, t $\Rightarrow$ time.

positive direction



(a) velocity $= \frac{dH}{dt} = 10 - 3.72t$. vector

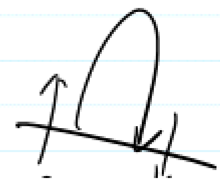
(a) velocity $= \frac{dH}{dt} = 10 - 3.72t.$ vector

$t = 1s$: velocity $= +6.28.\ (m/s)$.

(b) $t = a\,s$: velocity $:= 10 - 3.72a\ (m/s)$.

(c) $H = 0$

$10t - 1.86t^2 = 0 \Rightarrow t = 0s$



$10t - 1.86t^2 = 0 \Rightarrow t = 0s$

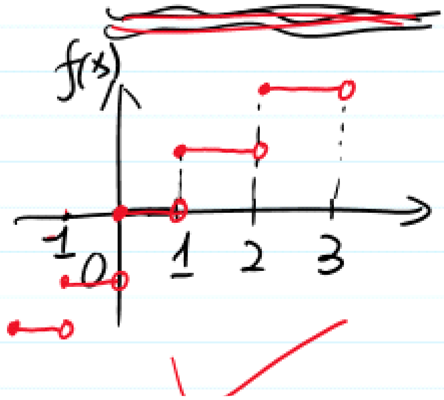
$t = \frac{10}{1.86} \approx 5.38\,s$

(d) $t = \frac{10}{1.86}$ s.

velocity: $\left.\frac{dH}{dt}\right|_{t=\frac{10}{1.86}} = -10\ (m/s)$

7. Greatest integer function: "floor of x"

$f(x) = [x]$. "that is less than or equal to x"

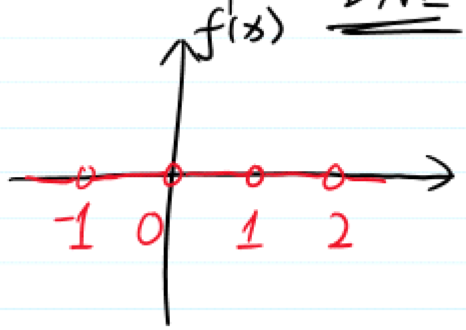


integer less than/equal to

$\Rightarrow 0$ $\frac{1}{3}$ "$\frac{1}{2}$".

$\Rightarrow 0$

$$f'(x) = \begin{cases} 0, & \text{if } x \neq \text{integer.} \\ \text{DNE}, & \text{if } x = \text{integer. Discontinuous} \end{cases}$$



8. $$\begin{cases} f'_-(a) = \lim\limits_{h\to 0^-} \dfrac{f(a+h)-f(a)}{h} \\ f'_+(a) = \lim\limits_{h\to 0^+} \dfrac{f(a+h)-f(a)}{h}. \end{cases}$$

$f'(a)$ exists $\Longleftrightarrow$ $f'_-(a) = f'_+(a)$ exists

(a)

$$f(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 5-x & \text{if } 0 < x < 4 \\ \dfrac{1}{5-x} & \text{if } x \ge 4. \end{cases}$$

$$f'_-(4) = \lim_{h\to 0^-} \frac{f(\overset{<4}{4+h}) - f(4)}{h} = \frac{5-(4+h)-1}{h}$$

$$f'_-(4) = \lim_{\substack{h \to 0^-}} \frac{f(4+h) - f(4)}{h} = \frac{5-(4+h)-1}{h}$$

$\Rightarrow \begin{cases} h<0 \\ h \to 0 \end{cases}$

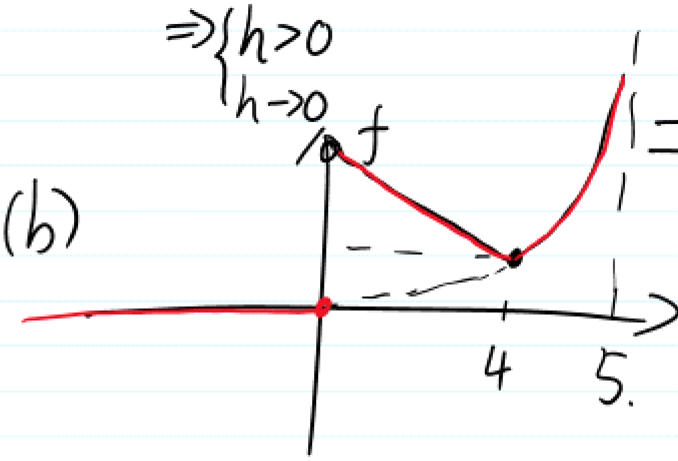
$$= \frac{-h}{h} = \underline{-1}.$$

$$f'_+(4) = \lim_{h \to 0^+} \frac{f(\underbrace{4+h}_{>4}) - f(4)}{h} = \frac{\frac{1}{5-(4+h)} - 1}{h}.$$

$\Rightarrow \begin{cases} h>0 \\ h \to 0 \end{cases}$

$$= \underline{1}.$$

(b)



(c) <u>Discontinous points</u>: $x=0$, $x=5$.

(d) NOT differentiable: $x=0.$, $x=5.$

$x=4$ <u>corner</u>?

$f'_-(4) \neq f'_+(4)$. f continous $x=4$

NOT differentiable $x=4$!

9. (a) The derivative of an even function is an odd function:

$f(-x) = f(x)$

$f'(-x) \overset{?}{=} -f'(x)$

$\Downarrow$

$$\lim_{h\to 0} \frac{f(-x+h)-f(-x)}{h} = -\lim_{h\to 0} \frac{f(x-h)-f(x)}{-h}$$

$$\overset{-h\to s}{=} -\lim_{s\to 0} \frac{f(x+s)-f(s)}{s}$$

$$= -f'(x).$$

11. $f(x) = \dfrac{x}{x+\frac{c}{x}}$ ($x \to f(x)$, $x+\frac{c}{x} \to g(x)$)

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}.$$

$$\Rightarrow \frac{d}{dx} f(x) = \frac{\left(x+\frac{c}{x}\right) - x\left(1-\frac{c}{x^2}\right)}{\left(x+\frac{c}{x}\right)^2}$$

$$\left(x+\frac{c}{x}\right)' = 1-\frac{c}{x^2}$$

$2c$

$$= \frac{\cancel{x} + \frac{c}{x} - \cancel{x} + \frac{c}{x}}{\left(x + \frac{c}{x}\right)^2} = \frac{\frac{2c}{x}}{\left(x + \frac{c}{x}\right)^2}.$$

12、 $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0.$

$$\Downarrow$$

$$p'(x) = a_n n x^{n-1} + a_{n-1}(n-1) x^{n-2} + \cdots + a_2 2x + a_1$$

$$= \sum_{i=1}^{n} i\, a_i\, x^{i-1}$$

15、 $f(x) = \sqrt{x} \cdot g(x).$

where $g(4) = 8$, $g'(4) = 7$,

find $f'(4)$

$$f'(x) = (\sqrt{x})' g(x) + \sqrt{x}\, g'(x).$$

$$= \frac{1}{2\sqrt{x}} g(x) + \sqrt{x}\, g'(x)$$

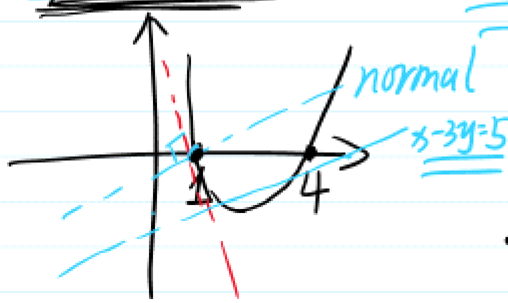
$$\Rightarrow f'(4) = \frac{1}{2\sqrt{4}}\, g(\underset{8}{\underset{\parallel}{4}}) + \sqrt{4}\; \underset{7}{\underset{\parallel}{g'(4)}}$$

$$= 16.$$

Self-practice #6

1. $y = x^2 - 5x + 4 = (x-1)(x-4)$ find the normal line that parallel to the $x - 3y = 5$ $\Rightarrow y = \frac{x-5}{3}$.

Pf:



Tangent line. $(5,0)$ $(0,-\frac{5}{3})$

$\begin{cases} ① \text{ point} \\ ② \text{ slope} \end{cases}$ $y' = \frac{dy}{dx}$.

$y = \frac{x-5}{3} = \frac{1}{3}x - \frac{5}{3}$

$\Rightarrow$ Slope of normal line is $\frac{1}{3}$ ①

$\Rightarrow$ $\boxed{2D}$: Slope of tangent line $\times$ slope of normal line $= -1$

same point

$\Rightarrow$ slope of tangent line: -3

$y' = \frac{dy}{dx} = 2x - 5 =$

$y' = \frac{dy}{dx} = 2x - 5$

$y' = \frac{dy}{dx} = 2x - 5$

$2x - 5 = -3$ ②

$\Rightarrow x = 1. \Rightarrow y = y = 0.$

By ① and ②.

$\Rightarrow$ normal line: $y - 0 = \frac{1}{3}(x-1)$

$\Rightarrow y = \frac{x-1}{3}.$

7. (a) $f(x) = \frac{\tan x - 1}{\sec x}$

$\xRightarrow[\text{Rule}]{\text{Quotient}} f'(x) = \frac{(\tan x - 1)' \cdot \sec x - (\tan x - 1) \cdot (\sec x)'}{(\sec x)^2}.$

$= \frac{\tan x + 1}{\sec x}.$ ✓

$(\tan x - 1)'$

$= (\tan x)' = \left(\frac{\sin x}{\cos x}\right)'$

$= ?$

$(\sec x)'$

$= \left(\frac{1}{\cos x}\right)' = ?$

(b) Simplify

$f(x) = \frac{\tan x - 1}{\sec x} = \frac{\frac{\sin x}{\cos x} - \frac{\cos x}{\cos x}}{\frac{1}{\cos x}}$

$= \sin x - \cos x.$

$f'(x) = \cos x + \sin x.$ ✓

$f(x) = \cos x + \sin x.$ ✓

(c) $\dfrac{\tan x + 1}{\sec x} \overset{?}{=} \cos x + \sin x.$

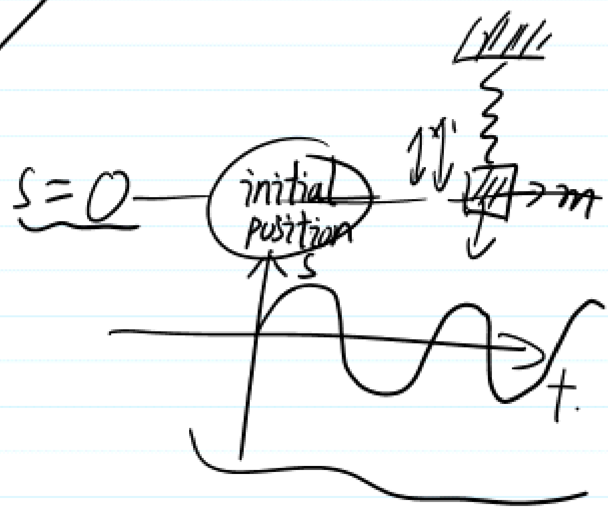
① ③ ②

$= \dfrac{\frac{\sin x}{\cancel{\cos x}} + \frac{\cos x}{\cancel{\cos x}}}{\frac{1}{\cancel{\cos x}}}$

14. $\boxed{s = A\cos(\omega t + \delta)}$

distance

time.



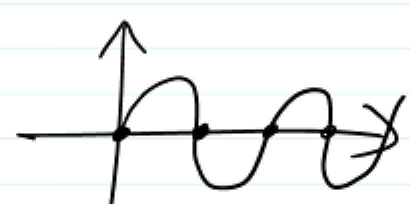
(a) velocity $= \dfrac{ds}{dt} \overset{\text{chain}}{\underset{\text{rule}}{=}} A(-\sin(\omega t + \delta))\underbrace{(\omega t+\delta)'}_{=\omega}$

$= -A\omega \sin(\omega t + \delta)$

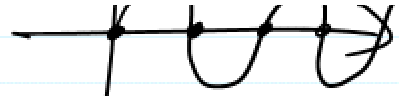
(b) when is velocity $= 0$?

$-A\omega \sin(\omega t + \delta) = 0$

$\Rightarrow \sin(\omega t + \delta) = 0.$



$\Rightarrow \sin(\omega t+\delta)=0.$



$\Rightarrow \omega t+\delta = n\pi,\ n=\dots,-2,-1,0,1,2\dots$ $\quad n\pi,\ n=0,1,2\dots\ \dots-2,-1$

$= 0, \pm 1, \pm 2, \dots$

$\Rightarrow t = \frac{n\pi-\delta}{\omega},\ n=0,\pm1,\pm2,\dots$

Assignment 1

Friday, October 16, 2020 11:37 AM

8. Use precise definition to show $\lim_{x\to1}(x^4-1)=0$

① "δ-ε" language

Def: $\forall \varepsilon>0,\ \exists \delta>0$, when $|x-1|<\delta$,

then $\cancel{|f(x)-f(x_0)|}$ $\quad \delta(\varepsilon)$

$|x^4-1\ \cancel{0}\ |<\varepsilon.$

② Thoughts: reverse process "$x\in\mathbb{R}$"

$|x^4-1|<\varepsilon.$ "local definition"

$|x^4 - 1| < \varepsilon$

$\Downarrow$

$|(x-1)(x^3+x^2+x+1)| < \varepsilon$

"local definition"

"$\delta' = 1$" $|x-1| < \delta'$

$|x-1| < 1$

$\Rightarrow 0 < x < 2$

$\Downarrow \quad \Downarrow$

$|\delta| \cdot |x^3+x^2+x+1| < \varepsilon$

$\Downarrow \leq 15$

upper bounder / maximum.

$\Rightarrow \delta < \dfrac{\varepsilon}{15}$

$= 1$

$\delta < \dfrac{\varepsilon}{15}$

③ $\forall \varepsilon > 0$, $\exists \delta = \min\{\frac{\varepsilon}{15}, \delta'\}$, when $|x-1| < \delta$

then $|x^4 - 1| = \underbrace{|x-1|}_{<\delta} \underbrace{|x^3+x^2+x+1|}_{<15}$

$< \varepsilon$

$\Rightarrow$ Conclude that $\lim\limits_{x \to 1} (x^4 - 1) = 0$ #

Assignment 2

Friday, October 23, 2020 11:41 AM

P6. If f continuous, is the $|f(x)|$ continuous?

$\boxed{\text{Yes}}$ prove. $\boxed{\text{No}}$, ~~counterexample~~

Pf: Given $f : A \to \mathbb{R}$ continuous at $c \in A$

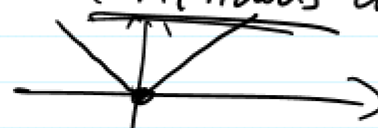
Given $f: \underline{A} \to \mathbb{R}$ continous at $c \in A$

$g: B \to \mathbb{R}$. continous at $f(c) \in B$.

$\Rightarrow \underbrace{g \circ f = g(f(x))}_{\text{well-defined}}$ continous at c.

Let $\boxed{f(x)} = y$, $|y| \leftarrow |x| \quad x \in \mathbb{R}$

① continous ② continuous as well



$x=0$

$\Rightarrow |f(x)|$ is continuous at .

P7: "IVT"

A fixed point of f is number c

$\boxed{f(c) = c} \Rightarrow$ $x \in$ Domain $f(x) = x$

$f(x) = x$

"IVT": ~~prove~~ any continuous function $f(x)$ domain $[0,1]$, and rang $[0,1]$. must have such fixed point.

and rang $[0,1]$. must have such fixed point.

Pf: $G(x) = f(x) - x$ continuous (A) $[0,1]$.

$\Leftrightarrow$ $G(x)$ has a zeros

$\Leftrightarrow$ $G(x) = 0$ has a root.

(B) $\begin{cases} G(0) = f(0) - 0 = f(0) \geq 0. \\ G(1) = f(1) - 1 \leq 0. \end{cases}$

$0 \leq \quad \leq 1$

(A)+(B) $\underset{\text{IVT.}}{\Longrightarrow}$ conclude that $f(c) = c$

$G(c) = 0 \Longleftrightarrow$ $\Updownarrow$

$c \in [0,1]$

Assignment 3

Assignment 3

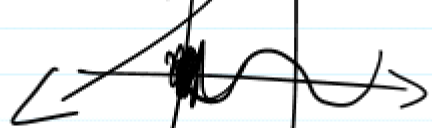
Friday, November 13, 2020 11:46 AM

1、 $f(x) = \begin{cases} x\cos\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$ $\Rightarrow f(x) = \begin{cases} - & x \le 0 \\ - & x \ge 0 \end{cases}$

Definition :

$$\lim_{\substack{x \to 0 \\ \{x \neq 0}} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{\cancel{x} \cos\left(\frac{1}{x}\right) - 0}{\cancel{x}} = \lim_{x \to 0} \cos\left(\frac{1}{x}\right)$$

$-1 \le \quad \le 1$



divergent

$\Rightarrow f(x)$ is NOT differentiable.

$\begin{cases} \lim_{x \to 0^-} \\ \lim_{x \to 0^+} \end{cases}$

$$\lim_{x \to 0} \boxed{x \cos\left(\frac{1}{x}\right)} = 0$$

$-|x| \le \qquad \le |x|$

$\to 0 \qquad \to 0.$

2、 $f(x) = x^3 + 5x + 4$. $x \in \mathbb{R}$. set $c \in \mathbb{R}$

$\Rightarrow \boxed{f'(x) = 3x^2 + 5}$ by first principle

$\Rightarrow$ Definite

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c} \frac{x^3 + 5x + \cancel{4} - c^3 - 5c - \cancel{4}}{x - c}$$

$$= \lim_{x \to c} \frac{x^3 - c^3 + 5(x - c)}{x - c}$$

$x \to c$

$= \lim\limits_{x \to c} \dfrac{\cdots}{x-c}$ $\quad x \to c$

$= \lim\limits_{x \to c} \dfrac{\cancel{(x-c)}(x^2+xc+c^2+5)}{\cancel{x-c}}$

$= \lim\limits_{x \to} x^2+xc+c^2+5$

$= c^2+c^2+c^2+5$

$= \boxed{3c^2+5.}$

4. $y = \boxed{\sqrt{4+4\sin x}}$ $\quad y = \dfrac{dy}{dx}$

tangent line $\quad \boxed{(0,2)}$ $\quad$ normal line.

$\Downarrow$ $\boxed{y' = \dfrac{dy}{dx}}$ $\quad \boxed{\text{"2D"}}$ $\quad \Downarrow$

$\Downarrow$

Slope of tangent line $\times$ Slope of normal line $= -1$

$\Downarrow$ $\quad \Downarrow$

$\dfrac{dy}{dx} = \dfrac{1}{2}(4+4\sin x)^{\frac{1}{2}-1} \cdot (4+4\sin x)'$ $\quad$ slope normal line $= -1$.

$= \dfrac{1}{2} \dfrac{1}{\sqrt{4+4\sin x}} \cdot 4\cos x$ $\quad y-2 = -1(x-0)$

$= \dfrac{2\cos x}{\sqrt{4+4\sin x}}.$ $\quad \Rightarrow y = -x+2.$ #

$\sqrt{4+4\sin 7x}$.

$\Rightarrow y = -x+2$. #.

slope $= y'(0,2) = 1$.

$\Rightarrow y-2 = 1(x-0)$

$\Rightarrow y = x+2$.

$\boxed{6.}$ Leibniz rule: general product rule

$(fg)' = f'g + fg'$

$$(fg)^{(n)} = \sum_{k=0}^{n} C_k^n f^{(n-k)} g^{(k)}$$

$C_k^n = \frac{n!}{k!(n-k)!}$

Pf: Induction

$\Rightarrow$ ① Base step: $\boxed{n=1}$ product rule!

② Induction hypothesis:

$$(fg)^{(n)} = \sum_{k=0}^{n} C_k^n f^{(n-k)} g^{(k)}$$

$n+1$

$\Rightarrow$ Target: $(fg)^{(n+1)} = \sum C^{n+1} f^{(n+1-k)} g^{(k)}$

$\Longrightarrow$ Target: $(fg)^{(n+1)} = \sum_{k=0}^{n+1} C_k^{n+1} f^{(n+1-k)} g^{(k)}$

$$(fg)^{(n+1)} = \left(\sum_{k=0}^{n} C_k^n f^{(n-k)} g^{(k)} \right)'$$

$$= \left(C_0^n f^{(n)} g + C_1^n f^{(n-1)} g^{(1)} + \cdots + C_n^n f g^{(n)} \right)'$$

additive rule + product $= C_0^n \left(f^{(n+1)} g + f^{(n)} g^{(1)} \right) + \cdots + C_n^n \left(f^{(1)} g^{(n)} + f g^{(n+1)} \right)$

$$= \sum_{k=0}^{n} C_k^n f^{(n-k+1)} g^{(k)} + \sum_{k=0}^{n} C_k^n f^{(n-k)} g^{(k+1)}$$

select $\boxed{k=r}$: $C_r^n f^{(n-r+1)} g^{(r)}$ 1^{st}

$+$

select $\boxed{k=r-1}$: $C_{r-1}^n f^{(n-r+1)} g^{(r)}$ 2^{nd}

$$C_r^n + C_{r-1}^n = C_r^{n+1}$$

$$\Longrightarrow C_r^{n+1} f^{(n-r+1)} g^{(r)}$$

All in all: $(fg)^{(n+1)}$

$$= C_0^n f^{(n+1)} g + \sum^{n} C_r^{n+1} f^{(n-r+1)} g^{(r)} + C_n^n f g^{(n+1)}$$

$$= C_0^n f^{(n+1)} g + \sum_{r=1} C_r^{n+1} f^{(n-r+1)} g^{(r)} + C_n^n f g^{(n+1)}$$

$$C_0^n = 1 = C_0^{n+1} \qquad C_n^n = 1 = C_{n+1}^{n+1}$$

$$= \sum_{k=0}^{n+1} C_k^{n+1} f^{(n+1-k)} g^{(k)}$$

$\Rightarrow$ By induction, we prove that ...

7. $\frac{d^{30}}{dx^{30}}(\sin 2x) = ?$

7. $\frac{d^{30}}{dx^{30}}(\sin 2x) = ?$

$(\sin 2x)^{(n)}$

8. Suppose that $\lim_{x\to 0} (f(x) - g(x)) = 0$

$h(x)$ with $\mathbb{R}$. $\boxed{\delta = \varepsilon}$ $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$, if $|x-y| < \delta$

$|h(x) - h(y)| \le \varepsilon$.

$\boxed{a}$ Assume $\boxed{|h(x) - h(y)| \le \boxed{3}|x-y|}$ for any $x, y \in \mathbb{R}$. Select $\delta = \frac{\varepsilon}{3}$.

$|h(x) - h(y)| \le 3|x-y| \le 3 \cdot \frac{\varepsilon}{3} \le \varepsilon$.

$\Rightarrow$ $h(x)$ continuous. / stronger

$\overset{?}{\Rightarrow} \lim_{x\to 0} (h(f(x)) - h(g(x))) = 0$? ✓

Lipschitz continuous

$|h(f(x)) - h(g(x))| \le 3 |f(x) - g(x)|$

$$\lim_{x\to0} \ldots$$

$$0 \le \lim_{x\to0} |h(f(x)) - h(g(x))| \le 3 \lim_{x\to0} |f(x) - g(x)| \to 0.$$

Lipchitz continuous

by sandwich/squeezing theorem, $\lim_{x\to0} |h(f(x)) - h(g(x))| \to 0$

$\boxed{b)}$ Assume $h(x)$ continuous function

$\Rightarrow$ X.

$\boxed{h(x) = x^2}$ $\quad f(x) = \frac{1}{x}, \quad g(x) = \frac{1}{x} + x$

$$\lim_{x\to0} (h(f(x)) - h(g(x))) \overset{?}{=} 0$$

$$= \lim_{x\to0} \left(\left(\frac{1}{x}\right)^2 - \left(\frac{1}{x} + x\right)^2 \right)$$

$$= \lim_{x\to0} \left(\left(\frac{1}{x}\right)^2 - \left(\frac{1}{x} + x\right)^2 \right)$$

$$= \lim_{x\to0} \left(\cancel{\frac{1}{x^2}} - \cancel{\frac{1}{x^2}} - 2 - x^2 \right)$$

$$= \lim_{x\to0} (-2 - x^2)$$

$$= -2 \neq 0.$$

$h(x) = x^2$ is NOT Lipschi...

$h(x) = x^2$ is NOT Lipschiz. continuous in $\mathbb{R}$

$|h(x)-h(y)| = |x^2-y^2|$

$= |x+y||x-y|$

$x, y \in \mathbb{R}$

$[0,1]$

Assignment 4

Saturday, November 21, 2020 10:20 PM

1. For the following two functions:
   (a) Find the intervals on which f is increasing or decreasing.
   (b) Find the local maximum and minimum values of f.
   (c) Find the intervals of concavity and the inflection points.
   (d) Sketch the curves with the information above

$$f(x) = \sin 2x - 2\cos x, \text{ for } 0 \le x \le 2\pi,$$

$$f(x) = \frac{x^2-4}{x^2-2x}, \text{ for } x \neq 0, 2.$$

$f(x) = \sin 2x - 2\cos x$, for $0 \le x \le 2\pi$,

$f(x) = \frac{x^2-4}{x^2-2x}$, for $x \neq 0, 2$.

Pf: (a) If $f(x) = \sin 2x - 2\cos x \Rightarrow f'(x) = 2\cos 2x + 2\sin x$

$\Rightarrow f'(x) = 0$

$2\cos 2x + 2\sin x = 0$

$\Rightarrow 2(1 - 2\sin^2 x) + 2\sin x = 0$

$\overset{y=\sin x}{\Longrightarrow} -4y^2 + 2y + 2 = 0$

$\Rightarrow -2y^2 + y + 1 = 0$

$\Rightarrow (2y+1)(-y+1) = 0$

$\Rightarrow y_1 = -\frac{1}{2}$ or $y_2 = 1$

$\Rightarrow \sin x = -\frac{1}{2}$ or 1 and $x \in [0, 2\pi]$

$\Rightarrow x = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{\pi}{2}$.

x	0	$(0, \frac{\pi}{2})$	$\frac{\pi}{2}$	$(\frac{\pi}{2}, \frac{7\pi}{6})$	$\frac{7\pi}{6}$	$(\frac{7\pi}{6}, \frac{11\pi}{6})$	$\frac{11\pi}{6}$	$(\frac{11\pi}{6}, 2\pi)$	2π
$f'(x)$	>0	>0	0	>0	0	<0	0	>0	>0
$f(x)$		$\nearrow$	?	$\nearrow$	Max	$\searrow$	Min	$\nearrow$	

$\therefore$ Local $\begin{cases} \text{maximum } x = \frac{7\pi}{6} \\ \text{minimum } x = \frac{11\pi}{6}. \end{cases}$

$0 \quad \frac{\pi}{2} \quad \pi \quad \frac{7\pi}{6} \quad \frac{11\pi}{6} \quad 2\pi$

(b) $f(x) = \sin 2x - 2\cos x \Rightarrow f''(x) = -4\sin 2x + 2\cos x \Rightarrow f''(x) = 0$

$\Rightarrow -4 \cdot 2\sin x\cos x + 2\cos x = 0$

$P_1 = \frac{2\pi}{45}$ $\quad P_2 = \frac{\pi}{3}$ $\quad P_3 = \frac{32\pi}{45}$ $\quad P_4 = \frac{5\pi}{4}$

$\Rightarrow \cos x(1 - 4\sin x) = 0$

$\Rightarrow \cos x = 0$ or $\sin x = \frac{1}{4}$

x	$(0, \arcsin\frac{1}{4})$	$(\arcsin\frac{1}{4}, \frac{\pi}{2})$	$(\frac{\pi}{2}, \pi - \arcsin\frac{1}{4})$	$(\pi - \arcsin\frac{1}{4}, \frac{3\pi}{2})$
$f''(x)$	>0	<0	>0	<0
$f(x)$	CU	CD	CU	CD

$\Rightarrow \begin{cases} x_1 = \frac{\pi}{2}, \quad x_2 = \frac{3\pi}{2}, \\ x_3, x_4 = \sin^{-1}(\frac{1}{4}) \end{cases}$ or $\begin{cases} x_3 = \arcsin\frac{1}{4} \end{cases}$

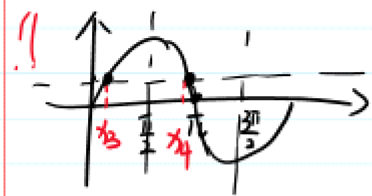
Inflection points

x^2 $\quad -x^2$

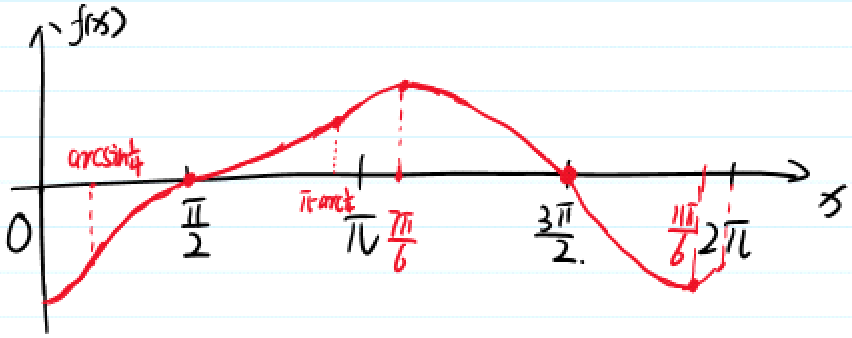
Inflection points $x_3, x_4 = \sin^{-1}(\frac{1}{4})$ or $\begin{cases} x_3 = \arcsin\frac{1}{4} \end{cases}$

$x_4 = \pi - \arcsin\frac{1}{4}$.

$x_4 = \pi - \arcsin\frac{1}{4}$



(c) $f(x) = \sin 2x - 2\cos x$



when

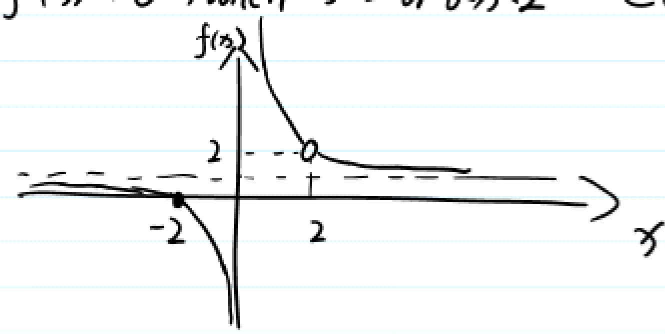
$f(x) = \frac{x^2-4}{x^2-2x}$, for $x \neq 0, 2$ $\Rightarrow (-\infty, 0) \cup (0, 2) \cup (2, +\infty)$

$= \frac{(x+2)(x-2)}{x(x-2)} = \frac{x+2}{x} = 1 + \frac{2}{x}$

$f'(x) = -\frac{2}{x^2} = 0$ No solution $\Rightarrow$ No local min/max

$f''(x) = \frac{4}{x^3} \neq 0$ for all $x \in \mathbb{R}$ $\Rightarrow$ No inflection points

$\begin{cases} f''(x) < 0, \text{ when } x < 0 & CD \\ f''(x) > 0, \text{ when } x > 2 \text{ or } 0 < x < 2 & CU \end{cases}$



$\left\{ \begin{array}{l} \left\{ \begin{array}{l} \text{asymptote line} \\ \text{intercept points} \end{array} \right. \\ \text{key points} \end{array} \right.$

2. Prove that, for all $x > 1$,

$$2\sqrt{x} > 3 - \frac{1}{x}.$$

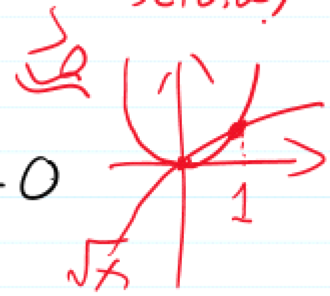
Pf: let $f(x) = 2\sqrt{x} - (3 - \frac{1}{x}) = 2\sqrt{x} + \frac{1}{x} + 3$.

Domain. $x \in (0, \infty)$

Domain. $x \in (0, \infty)$

Pf: let $f(x) = 2\sqrt{x} - (3 - \frac{1}{x}) = 2\sqrt{x} + \frac{1}{x} + 3.$

$f'(x) = \frac{1}{\sqrt{x}} - \frac{1}{x^2} = \frac{x^2 - x^{\frac{1}{2}}}{x^{\frac{5}{2}}}$



$f'(x) > 0$, if $x > 1$ and $f(1) = 0$

$\Rightarrow f(x) > 0$ for all $x > 1$.

$\Rightarrow 2\sqrt{x} > 3 - \frac{1}{x}$. #

3. The volume of a cube is increasing at a rate of $10cm^3/min$. How fast is the surface area increasing when the length of an edge is $30cm$?

rate

Pf: set the length of cube edge l cm.

$\Rightarrow \boxed{V = l^3} \rightarrow l(t)$ when $\frac{dV}{dt} = 10$, $l = 30$

$\xrightarrow[\text{w.r.t } t]{\text{differentiate}}$ $\frac{dV}{dt} = 3l^2 \frac{dl}{dt}$ $\quad \frac{dl}{dt} = \frac{10}{3 \times 30^2} = \frac{1}{270}$

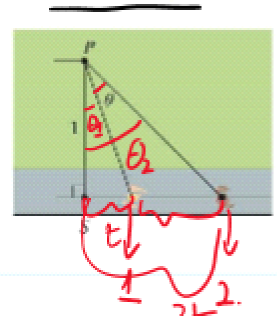
$\Downarrow$

surface area $\boxed{S = 6l^2.}$

$\Downarrow$

$\frac{dS}{dt} = 6 \cdot 2l \cdot \frac{dl}{dt} = 12 \times 30 \times \frac{1}{270} = \frac{4}{3}.$

4. An observer stands at a point P, one unit away from a track. Two runners start at the point S in the figure and run along the track. One runner runs three times as fast as the other. Find the maximum value of the observer's angle of sight between the runners.



$\theta = \theta_2 - \theta_1.$

If slow one run t units, then faster $3t$ units

$\theta_1 = \tan^{-1}(t)$, $\theta_2 = \tan^{-1}(3t)$

$\theta_1 = \tan^{-1}(t)$, $\theta_2 = \tan^{-1}(3t)$

$$\Rightarrow \theta(t) = \tan^{-1}(3t) - \tan^{-1}(t)$$

$$\Rightarrow \theta'(t) = \frac{3}{1+(3t)^2} - \frac{1}{1+t^2} = \frac{-6t^2+2}{(1+t^2)(1+9t^2)}$$

Then $\theta'(t) = 0 \Rightarrow t = \frac{\sqrt{3}}{3}$ or $-\frac{\sqrt{3}}{3}$ (reject)

$\Downarrow$

maximum.

$\nearrow \quad \searrow$

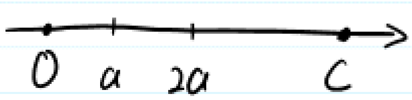
$$\theta = \tan^{-1}(\sqrt{3}) - \tan^{-1}\left(\frac{\sqrt{3}}{3}\right) = \frac{\pi}{6}.$$

5. If f is a differentiable function on $[0, c]$ such that $f(0) = 0$ and $\boxed{f'(x) \text{ is increasing}}$, prove that for any a satisfying the restriction there holds

$$0 < a < 2a < c,$$
$$2f(a) \le f(2a).$$

Pf: By Mean Value Theorem $\begin{cases} f \text{ differentiable on } (a,b) \\ f \text{ continuous } [a,b]. \end{cases}$

$\xRightarrow{\text{Then}}$ there exist $c \in (a,b)$, such that $f'(c) = \frac{f(b)-f(a)}{b-a}.$



MVT for $(0, a)$

$\exists\, c_1 \in (0,a)$, s.t. $f'(c_1) = \frac{f(a)-f(0)}{a-0} = \frac{f(a)}{a}.$

MVT for $(a, 2a)$

$\exists\, c_2 \in (a, 2a)$, $f'(c_2) = \frac{f(2a)-f(a)}{2a-a} = \frac{f(2a)-f(a)}{a}.$

$\Rightarrow f'(x)$ increasing $\Rightarrow f'(c_2) \ge f'(c_1)$

$$\Rightarrow f(2a) \ge 2f(a).$$

$\Rightarrow f(2a) \geq 2f(a)$.

6. Suppose that a function f is continuous on $[a, b]$ ① and $f''(x)$ exists for every $x \in (a, b)$ ②. If $f(a) = f(b) = f((a+b)/2)$, prove that there exists some $\zeta \in (a, b)$ such that $f''(\zeta) = 0$.

Pf: $f''(x)$ exists for $x \in (a,b) \Rightarrow f'(x)$ exist $\Rightarrow$ $f(x)$ differentiable for $x \in (a,b)$

① and ② ⊕ $f(a) = f(b)$ by Rolle's Theorem $\Downarrow$ $\exists\, c \in (a,b)$, s.t. $f'(c) = 0$.

① and ② ⊕ $f(a) = f(\frac{a+b}{2})$ $\xRightarrow{\text{Rolle's}}$ $\exists\, z_1 \in (a, \frac{a+b}{2})$ $f'(z_1) = 0$.

① and ② ⊕ $f(\frac{a+b}{2}) = f(b)$ $\xRightarrow{\text{Rolle's}}$ $\exists\, z_2 \in (\frac{a+b}{2}, b)$ $f'(z_2) = 0$.

$\xRightarrow{\text{Rolle's}}$ $\exists\, z \in (z_1, z_2) \subset (a,b)$ $f''(z) = 0$

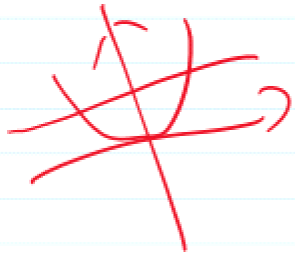
7. Using the definition of log function, prove that $\boxed{\log_b(a) = \frac{\ln a}{\ln b}}$.

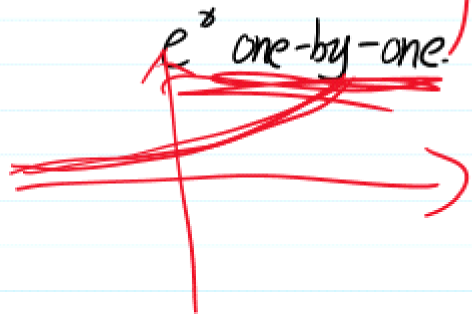
Pf: $z = \log_b a \Rightarrow b^z = a$. $\Rightarrow e^{b^z} = e^a$

$$\left\{\begin{array}{l} a = e^{\ln a} \\ b^z = e^{\ln b^z} = e^{z \ln b} \end{array}\right\} \Rightarrow e^{\ln a} = e^{z \ln b} \Rightarrow \ln a = z \ln b$$

$b^z = e^{\ln b} = e^{z \ln b}$ ⊕ $\Rightarrow \ln a = z \ln b$

$\Downarrow$





$z = \frac{\ln a}{\ln b}$. #

Midterm Exam

Saturday, October 31, 2020 3:47 PM

MA1300 Tutorial Class $\boxed{TB}$ session.

Tutor: QI Kunlun

E-mail: kunlun.qi@my.cityu.edu.hk

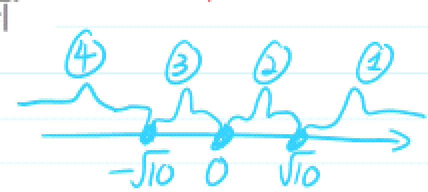
Office: Room 1392, FYW Building

1. (12 points) Find the (largest possible) domain of the function

$$F(x) = \frac{1}{\sqrt{|x^2 - 10| - 3|x|}}.$$

$\begin{cases} ① \neq 0 \\ ② \geq 0 \end{cases}$

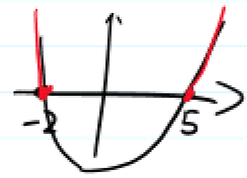
Pf: $|x^2 - 10| - 3|x| > 0$ ✓



key point/watershep point: 0, $\sqrt{10}$, $-\sqrt{10}$.

key point / watershep point : 0 , $\sqrt{10}$, $-\sqrt{10}$.

① If $x \ge \sqrt{10}$, $\quad |x^2-10-3x| > 0$



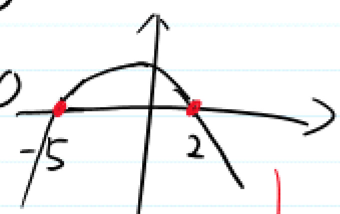
$\Rightarrow (x+2)(x-5) > 0$

$\Rightarrow x > 5$ or $x < -2$ (rejected)

Thus, $\boxed{x>5}$

② If $0 \le x < \sqrt{10}$, $\quad -x^2+10-3x > 0$

$\Rightarrow -(x-2)(x+5) > 0$



$\Rightarrow -5 < x < 2$

Thus, $\boxed{0 \le x < 2}$

Similarly, discuss the $(-\infty, -\sqrt{10})$ and $(-\sqrt{10}, 0)$

Similarly, discuss the $\underbrace{(-\infty, -\sqrt{10})}_{③}$ and $\underbrace{(-\sqrt{10}, 0)}_{④}$

Or a trick: Since $|x^2-10|-3|x|$ even function

by symmetry, $\overset{④}{(-\infty,-5)}$ and $\overset{③}{(-2,0]}$ is valid.

$\Rightarrow$ Finally, the (largest possible) domain "set"!!!

$(-\infty,-5) \cup (-2,2) \cup (5,\infty)$ #

$\{x \mid \ldots < x < \ldots\}$

2. (12 points) Use the definition of derivative to determine if the following function $f(x)$ is differentiable at $x = 0$.

$$f(x) = \begin{cases} x^2\cos(1/x) & \text{if } x > 0, \\ x\sin(x) & \text{if } x \le 0. \end{cases} \Leftarrow \text{"} x=0 \text{"}$$

Pf: $\boxed{\text{Traget}}$: left derivative $\neq$ right derivative

Pf: $\boxed{\text{Traget}}$: left derivative $\overset{?}{=}$ right derivative

$$\Rightarrow f'_+(0) \overset{?}{=} f'_-(0)$$

$$f'_+(0) = \lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{h^2\cos(\frac{1}{h})-0}{\cancel{h}} = \lim_{h\to 0^+} h\cos(\frac{1}{h})$$

$\begin{cases} h>0 \\ h\to 0\end{cases}$

"squeeze Th" $\Rightarrow 0$

$$f'_-(0) = \lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{\cancel{h}\sin(h)}{\cancel{h}} = \lim_{h\to 0^-} \sin(h)$$

$\begin{cases} h<0 \\ h\to 0\end{cases}$

$\Rightarrow 0.$

Thus, $f'_+(0) = f'_-(0) \Rightarrow f'(0)=0$ exists. #

3. a) (14 points) Use the $\boxed{\text{precise definition}}$ of limit to show that

$$\lim_{x\to 1} \underbrace{\frac{x^2+2x-3}{x^2-1}}_{f(x)} = 2.$$

"$\delta-\varepsilon$"

"local"

Pf: Reminder: $\forall \varepsilon>0$, $\exists$ "δ" >0, such that

if $|x-x_0|<\delta$, "then" $|f(x)-f(x_0)|<\varepsilon$.

if $|x-x_0|<\delta$, then $|f(x)-f(x_0)|<\varepsilon$.

Target: $\forall \varepsilon>0$, $\exists \overset{\text{"}\delta(\varepsilon)\text{"}}{\delta} >0$, such that, if $|x-1|<\delta$,

$$\left|\frac{x^2+2x-3}{x^2-1}-2\right|<\varepsilon$$

$$\Rightarrow \left|\frac{(x+3)(x-1)}{(x+1)(x-1)}-2\right|<\varepsilon$$

$$\Rightarrow \left|\frac{x+3}{x+1} - 2\frac{(x+1)}{(x+1)}\right| < \varepsilon$$

if select a rough $\delta'=1$,

i.e. $|x-1|<\delta' \Rightarrow 0<x<2$

then $1<|x+1|<3$

$$\Rightarrow \frac{|-x+1|}{|x+1|}<\varepsilon$$

$\Rightarrow \left| \frac{x+3}{x+1} - 2 \right| < \varepsilon.$ ($\frac{2(x+1)}{(x+1)}$)

$\Rightarrow \left| \frac{-x+1}{x+1} \right| < \varepsilon.$

$\Rightarrow \frac{|-x+1|}{|x+1|} < \varepsilon.$ ("δ")

$\Rightarrow \frac{|-x+1|}{|x+1|} < \varepsilon$ ($<\varepsilon$)

"Thoughts"

$\Rightarrow |-x+1| < \varepsilon$ ($<\delta<\varepsilon$)

So if we select "δ" $= \varepsilon$.

δ'

Solution: For any ($\forall \varepsilon > 0$) $\varepsilon > 0$. there exists $\delta = \min\{1, \varepsilon\}$ ("δ'"), such that $0 < |x-1| < \delta$, then

$$\left| \frac{x^2+2x-3}{x^2-1} - 2 \right| = \left| \frac{x+3}{x+1} - 2 \right| = \frac{|-x+1|}{|x+1|} < \delta < \varepsilon.$$

Thus, by definition, we prove

$$\lim_{x \to 1} \frac{x^2+2x-3}{x^2-1} = 2 \quad \#.$$

b) (14 points) Use the precise definition of limit to show that

$0 < |x - x_0| < \delta$

$$\lim_{x \to 1^-} \frac{\cos(x^2)}{x-1} \text{ does not exist.}$$

(>0, $x^2 \to 1^+$; $x<1$, $x>1$; <0, $\to 0$) $<$ $+\infty$ / $-\infty$

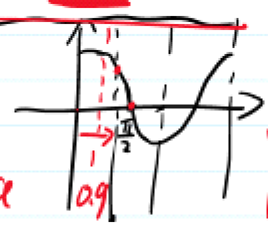
Pf: Reminder: $\forall M > 0$, $\exists \delta > 0$, such that if $-\delta < x - x_0 < 0$ (x_0 "1"), then $f(x) < -M$.

~~$\lim_{x \to x_0^-} f(x) = -M$~~

Target: $\forall M > 0$, $\exists \delta > 0$ ("δ" $\delta(M)$), such that. $-\delta < x - 1 < 0$,

Target: $\forall M > 0$, $\exists \delta > 0$ ($\delta(M)$), such that. $-\delta < x - 1 < 0$,

then $\frac{\cos(x^2)}{x-1} < -M$.



$\Rightarrow \; < \; \frac{\cos(0.81)}{}$, select a 0.9

Thoughts

$$\Rightarrow \quad < \frac{\cos(0.81)}{x-1}$$

$$\Rightarrow \quad < \frac{\cos(0.81)}{-\delta} < -M$$

$$\Rightarrow \quad \delta < \frac{\cos(0.81)}{M}$$

select a rough $\delta' = 0.1$. '0.2' "1" Toughts

$-1 < x-1 < 0$

$0 < x < 1$

$\Rightarrow -0.1 < x-1 < 0$

$\Rightarrow 0.9 < x < 1.$

Then $\cos(1) < \cos(x^2) < \cos(0.81)$

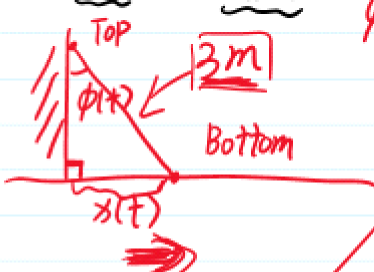
Rewrite: For $\forall M > 0$, $\exists$ "δ" $= \min\{0.1, \frac{\cos(0.81)}{M}\}$, such that if $-\delta < x-1 < 0$, then

$$\frac{\cos(x^2)}{x-1} \le \frac{\cos(0.81)}{x-1} < \frac{\cos(0.81)}{-\delta} < -M.$$

Thus $\Rightarrow$ by definition, we prove that

$$\lim_{x \to 1^-} \frac{\cos(x^2)}{x-1} = -\infty \quad \#.$$

$\boxed{4.}$ (12 points) A ladder 3m long rests against a vertical wall. Let $\phi(t)$ be the angle between the top of the ladder and the wall ($\phi \in [0, \pi/2]$) at time t and let $x(t)$ be the distance from the bottom of the ladder to the wall at time t. Assume that the bottom of the ladder slides away from the wall: ϕ and x are differentiable functions with respect to time t. When the angle ϕ increases at a rate of 0.1 rad/second and the distance x increases at a rate of 9 m/min, what is the value of the angle ϕ?



$\phi'(t) = 0.1$ $\qquad$ $x'(t) = 9$

$$\sin(\phi(t)) = \frac{x(t)}{3}$$

"Chain rule" $\Rightarrow$ differentiate

$$\cos(\phi(t)) \cdot \phi'(t) = \frac{1}{3} x'(t)$$

$$\Longrightarrow \cos(\phi(t)) \cdot 0.1 = \frac{1}{3} \times 9$$

$\Longrightarrow \cos(\phi(t))\cdot 0.1 = \frac{1}{3}\times 9$

Don't forget! $\Longrightarrow \cos(\phi(t)) = \frac{1}{2}$.

$\phi(t)\in[0,\frac{\pi}{2}]$ $\Longrightarrow \phi(t) = \cos^{-1}(\frac{1}{2}) = \frac{\pi}{3}$.

5. (12 points) If $f : [-1/2, 1/2] \to [0, 1]$ is a continuous function, prove that there is a number $c \in [-1/2, 1/2]$ such that $f(c) = \cos(\pi c)$.

"IVT"

Pf: Let $\boxed{F(x)} = f(x) - \cos(\pi x)$, continuous

$$\begin{cases} F(-0.5) = f(-0.5) \geq 0. \\ F(0) = f(0) - 1 \leq 0. \end{cases}$$

If $F(0) = 0$ or $F(-0.5) = 0$, Done.

If NOT, By IVT, there exist $c \in (-\frac{1}{2}, 0) \subset [-\frac{1}{2}, \frac{1}{2}]$,

such that $F(c) = 0 \Rightarrow f(c) = \cos(\pi c)$. #

6. (12 points) Let $f(x)$ and $g(x)$ be continuous functions at any $x \in \mathbb{R}$. Let $F(x) = \max\{f(x), g(x)\}$ for any $x \in \mathbb{R}$. Is the function $F(x)$ always continuous? If yes, provide a proof; if not, provide a counterexample.

Pf: $F(x) = \max\{f(x), g(x)\}$

$$= \frac{|f(x) - g(x)| + f(x) + g(x)}{2}$$ Tricky

check $\Longrightarrow$ $\begin{cases} \text{if } f(x) \geq g(x), = \dfrac{f(x) - g(x) + f(x) + g(x)}{2} = f(x) \\ \text{if } f(x) < g(x), = \dfrac{g(x) - f(x) + f(x) + g(x)}{2} = g(x) \end{cases}$

Since $|x|$ is continous. So are $f(x), g(x)$,

then after linear combination F(x) ...

Since f, g is continuous, so are $f(x), g(x)$,

then after linear combination, $F(x)$ is continuous as well

then after linear combination, $F(x)$ is continuous as well

#.

7. (12 points) If a function $\boxed{F(x) = f(g(x))}$ is differentiable at $x = 0$ and $f'(y)$ exists at $y = g(0)$. Given that $f'(g(0)) = 2$. Is the function $g(x)$ always differentiable at $x = 0$? If yes, provide a proof; if not, provide a counterexample.

NOT!

Chain rule

Pf: $\cancel{F'(x) = f'(g(x)) \cdot g'(x).}$

$\cancel{\Rightarrow F'(0) = f'(g(0)) \cdot g'(0)}$

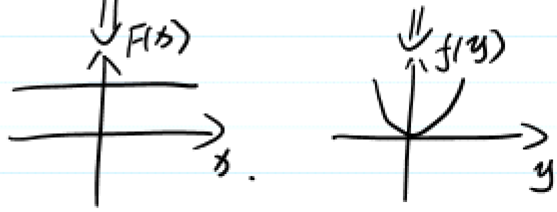
Valid, if Both $f(y), g(x)$ are differentiable!!!

NOT, counterexample:

let $f(y) = y^2$; let $g(x) = \begin{cases} 1, & \text{if } x \geq 0 \\ -1, & \text{if } x < 0. \end{cases}$

Hence, $g(x)$ is NOT differentiable at $x = 0$.

BUT, $F(x) = f(g(x))$ and $f(y)$ are differentiable, with



$f'(g(0)) = f'(1) = 1$.

$F(x) = f(g(x)) = [g(x)]^2$